5.1

Rates of Change

Learning objectives:

i) To define the average rate of change of a function over an interval and to understand the concept of instantaneous rate of change at a point.

Average speed:

First, we consider a familiar concept.

The **average speed** of a moving body over a time interval is the distance covered during the time interval divided by the length of the interval.

Example 1:

A rock falls from the top of a 50 m cliff.

Physical experiments show that a solid object dropped from the rest to fall freely near the surface of the earth will fall

$$y = 5t^2 \ m$$

during the first t sec.

i) Find the average speed:
   a. During the first 2 sec of fall.
   b. During the 1-sec interval between second 1 and second 2.

ii) Find the speed of the rock at $t = 1$ and $t = 2$ sec.

Solution:

The average speed of the rock during a given time interval is the change in distance Δy, divided by the length of the time interval Δt.

i) The average speed

a) For the first 2 sec: $\frac{\Delta y}{\Delta t} = \frac{5 \times 2^2 - 5 \times 0^2}{2 - 0} = 10 m / sec$

b) From second 1 to second 2:

$$\frac{\Delta y}{\Delta t} = \frac{5 \times 2^2 - 5 \times 1^2}{2 - 1} = 15 m / sec$$

ii) We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$ having length $\Delta t = h$ as

$$\frac{\Delta y}{\Delta t} = \frac{5(t_0 + h)^2 - 5t_0^2}{h}$$

We cannot use this formula to calculate the "*instantaneous*" speed at t_0 by substituting $h = 0$, because we cannot divide by 0. But we can use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$.

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
0.1	10.5	20.5
0.01	10.05	20.05
0.001	10.005	20.005
0.0001	10.0005	20.0005
↓	↓	↓
0	10	20

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 10 as the length of the interval decreases. This suggests that the rock is falling at a speed of 10 m/sec at $t_0 = 1$ sec. Similarly, the rock's speed at $t_0 = 2$ sec would appear to be 20 m/sec.

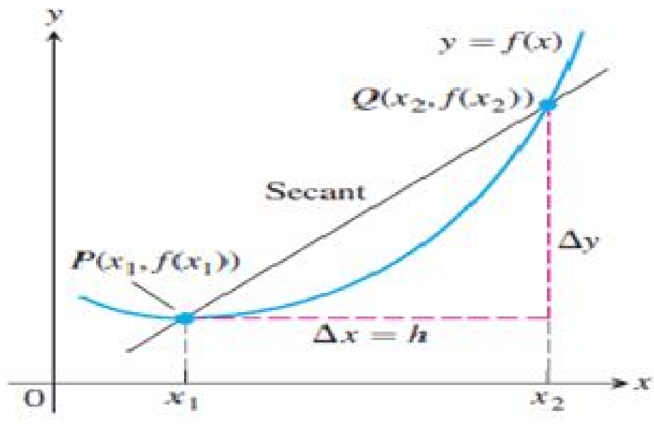
Average rate of change of a function:

Now, we introduce the concept of the ***average rate of change of a function***. Given a function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in value of y, $\Delta y = f(x_2) - f(x_1)$, by the length of the interval $\Delta x = x_2 - x_1 = h$ over which the change occurred.

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} \quad , h \neq 0$$

As seen from the figure below, the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$



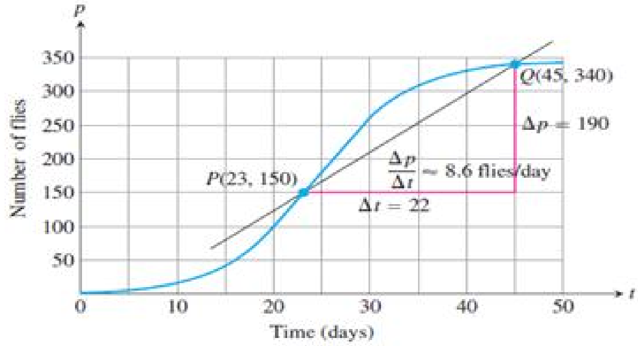
A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

In geometry, *a line joining two points of a curve is called a secant to the curve*. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of the secant PQ.

Example 2:

The figure below shows the growth of a population of flies in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve.

(a) Find the average growth rate from day 23 to day 45.
(b) How fast was the number of flies growing on day 23 itself?



Solution

(a) There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by 340 − 150 = 190 in 45 – 23 = 22 days. The average rate of change of the population p from day 23 to day 45 was

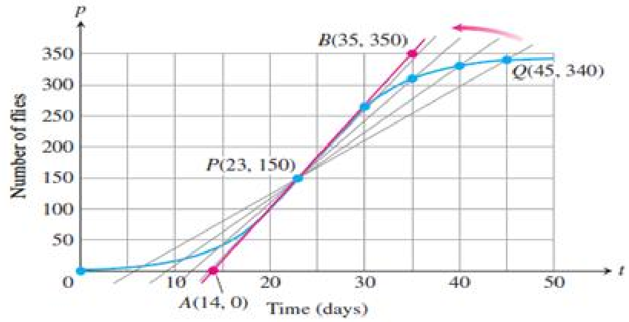
Average rate of change:

$$\frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies / day}$$

This average is the slope of the secant through the points P and Q on the graph.

(b) The average rate of change from day 23 to day 45, 8.6 flies/day, does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

We examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by *calculating the slopes of secants from P to Q for a sequence of points Q approaching P along the curve.*



The table below gives the positions of Q and slopes of four secants through the point P on the graph.

Q	Slope of PQ = Δp/Δt (flies/day)
(45,340)	$\frac{340-150}{45-23} \approx 8.6$
(40,330)	$\frac{330-150}{40-23} \approx 10.6$
(35,310)	$\frac{310-150}{35-23} \approx 13.3$
(30,265)	$\frac{265-150}{30-23} \approx 16.4$

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the *t*-coordinate of Q decreases from 45 to 30, and we would expect the slopes to continue rise higher as *t* continued on toward 23. Geometrically, the secants rotate about P and seem to approach the line PA in the figure, a line that goes through P in the same direction that the curve goes through P. This line is called the ***tangent*** to the curve at P. Since this line passes through the points (14,0) and (35,350), it has slope

$$\frac{350-0}{35-14} = 16.7 \ flies \ / \ day$$

On day 23 the population was increasing at a rate of about 16.7 flies/day.

The rate at which the rock in Example 1 was falling at instants $t = 1$ and $t = 2$ and the rate at which the population in Example 2 was changing on day $t = 23$ are called ***instantaneous rates of change***. As the examples suggest, we find instantaneous rates as limiting values of average rates. In Example 2, we also pictured the tangent line to the population curve on day 23 as a limiting position of secant lines.

P1:

Find the average rate of change of the function over the given interval

$f(x) = x^3+1; [-1,\ 1]$

Solution:

Given, $f(x) = x^3 + 1, x \in [-1,\ 1]$

Now $f(1) = 2, f(-1) = 0$

Average rate of change over $[-1,\ 1]$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$$

P2:

Find the average rate of change of the function over the given interval

$f(t) = cot\, t\, ; t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Solution:

Given, $f(t) = \cot t, t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$f\left(\frac{\pi}{4}\right) = cot\frac{\pi}{4} = 1,$$

$$f\left(\frac{3\pi}{4}\right) = cot\frac{3\pi}{4} = -1$$

Average rate of change

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = \frac{-4}{\pi}$$

P3:

The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in Rs. 1000
1990	6
1991	27
1992	62
1993	111
1994	174

What is the average rate of increase of the profit between 1992 and 1994?

Solution:

Profit in 1994 $= 174$

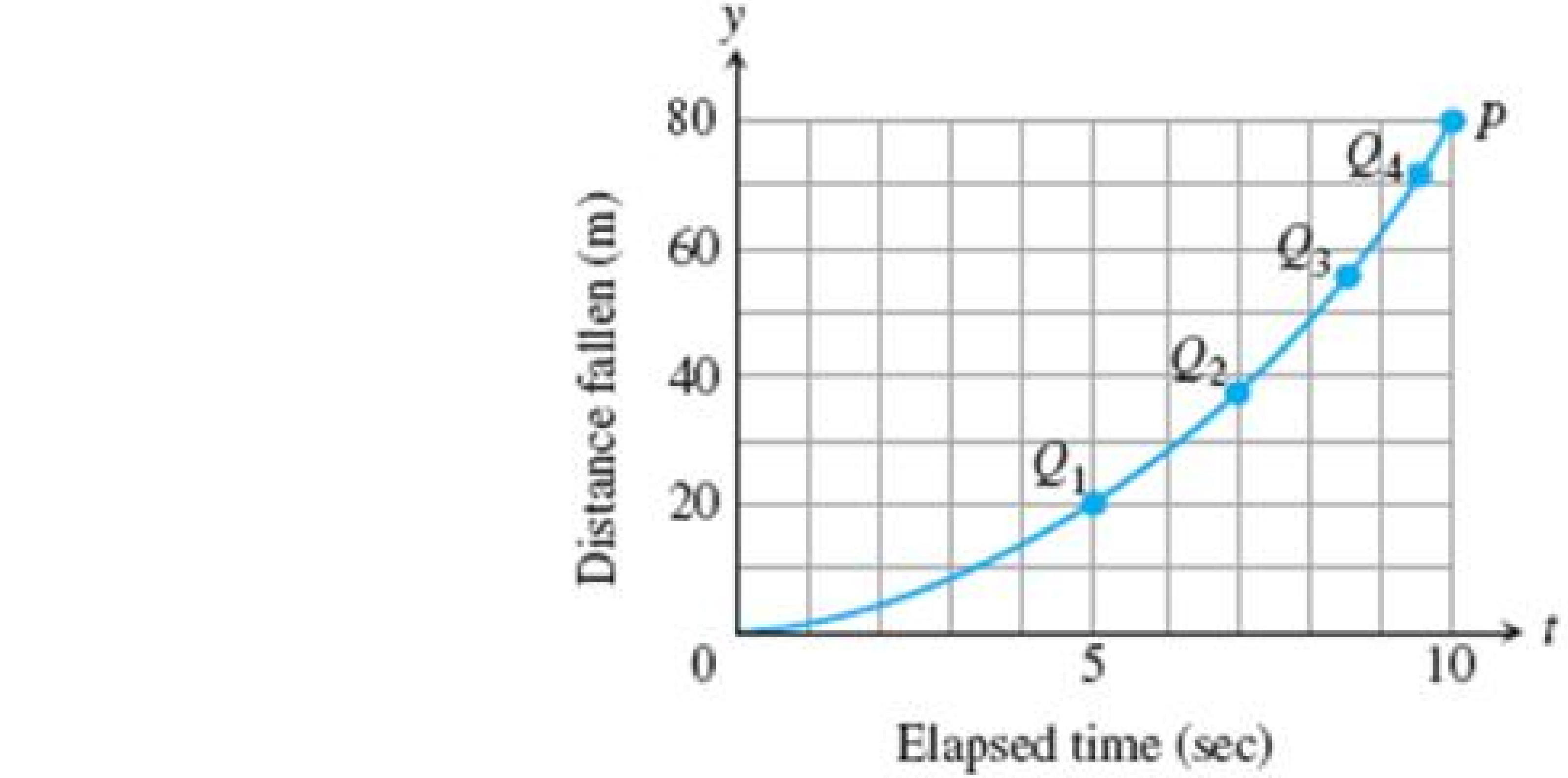
Profit in 1992 $= 62$

Average rate of increase of the profits $= \frac{174-62}{2} = 56$

The average rate of increase of the profits from 1992 to 1994 is Rs56,000.

P4:

The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80m to the surface of the moon.



Estimate the slopes of the secant PQ_1?

Solution:

Given , $P(10{,}80)$, $Q_1(5{,}20)$

Slope of the secant $PQ_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{80-20}{10-5} = \frac{60}{5} = 12$

IP1:

Find the average rate of change of the function over the given interval

$R = \sqrt{4\theta + 1}; [0,\ 2]$

Solution:

Step1:

Given, $R(\theta) = \sqrt{4\theta + 1}; \theta \in [0,\ 2]$

Step 2:

Now $R(2) = 3, R(0) = 1$

Step3:

We have

Average rate of change $= \frac{R(2)-R(0)}{2-0} = \frac{3-1}{2-0} = \frac{2}{2} = 1$

IP2:

Find the average rate of change of the function over the given interval

$f(t) = \tan t\,;\; t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Solution:

Step1:

Given, $f(t) = \tan t, t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Step2:

$$f\left(\frac{\pi}{4}\right) = tan\frac{\pi}{4} = 1,$$

$$f\left(\frac{3\pi}{4}\right) = tan\frac{3\pi}{4} = -1$$

Step3:

Average rate of change

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = \frac{-4}{\pi}$$

IP3:

The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in Rs. 1000
1990	6
1991	27
1992	62
1993	111
1994	174

What is the average rate of increase of the profits between 1990 and 1993?

Ans:

Step1:

Profit in 1993 $= 111$

Profit in 1990 $= 6$

Step2:

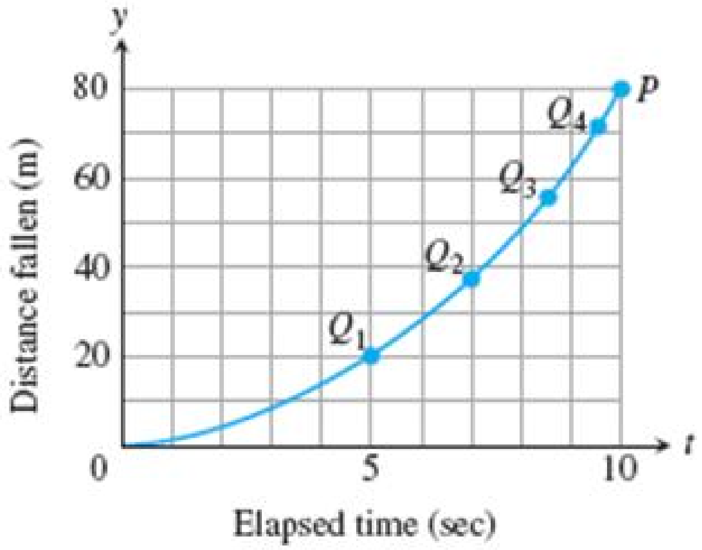
Average rate of increase of the profits $= \frac{111-6}{2} = \frac{105}{2} = 55.5$

Step3:

The average rate of increase of the profits from 1993 to 1990 is Rs55500.

IP4:

The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80m to the surface of the moon.



Estimate the slopes of the secant PQ_2?

Ans:

Step1:

Given , $P(10{,}80)$, $Q_2(7{,}40)$

Step2:

Slope of the secant $PQ_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{80-40}{10-7} = \frac{40}{3} = 13.33$

1) Find the average rates of change of the function over the given intervals

1.1) $f(x) = x^3 + 1$ $\quad [2,3]$

1.2) $g(x) = x^2$ $\quad [-1,1],[-2,0]$

1.3) $h(t) = \cot t$ $\quad [\frac{\pi}{6},\frac{\pi}{2}]$

1.4) $g(t) = 2 + \cos t$ $\quad [-\pi,\pi],[0,\pi]$

2) Find the average rates of change of the function over the given intervals

2.1) $(x) = \sqrt{x}\ ; x \geq 0\ , [1,2], [1,1.5], [1,1+h]$

2.2) $f(t) = \frac{1}{t}\ ; t \neq 0\ , [2,3], [2,T]$

5.2

Concept of Limit

Learning objectives

i) To understand the concept of the limit of a function through examples and to give an informal definition of the limit of a function

And

ii) To practice related problems.

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. First, we develop the limit intuitively and then formally.

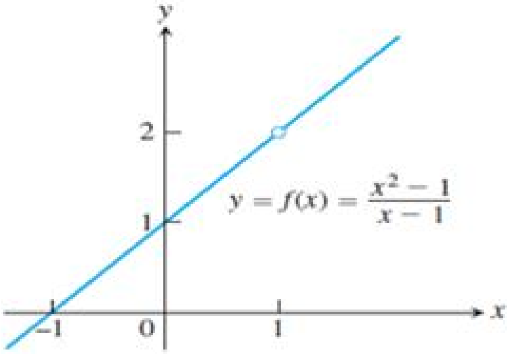
First we look at an example:

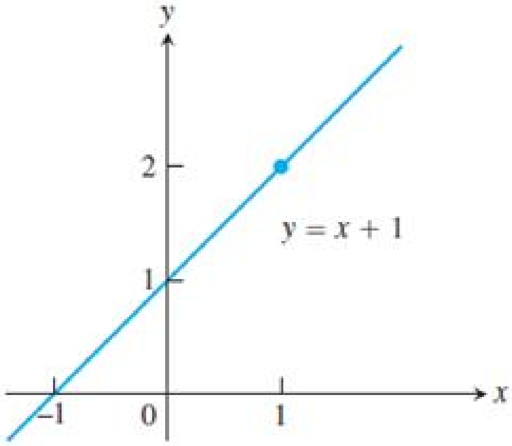
How does the function $f(x) = \frac{x^2 - 1}{x - 1}$ behave near $x = 1$?

The given function f is defined for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$ we can simplify the function by factoring the numerator and cancelling the common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for } x \neq 1$$

The graph of f is thus the line y = x + 1 with one point (1,2) removed. This removed point is shown as a “hole” in the figure.





The graph of f is identical with the line $y = x + 1$ except at $x = 1$ where f is not defined.

Even though *f*(1) is not defined, it is clear that we can make the value of *f*(*x*) *as close as we want* to 2 by choosing *x close enough* to 1.(see the following table)

Values of *x* Below and above 1	$\mathbf{f(x)} = \frac{\mathbf{x}^2 - 1}{\mathbf{x} - 1} = \mathbf{x} + 1, \quad \mathbf{x} \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We notice that the closer x gets to 1, the closer $f(x)$ seems to get 2.

We say that *f*(*x*) approaches arbitrarily close to 2 as *x* approaches 1, or, more simply, ***f(x) approaches the limit 2 as x approaches 1***. We write this as

$$\lim_{x \to 1} f(x) = 2 \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Definition

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself*. If there exists a real number L and *f*(*x*) gets arbitrarily close to L for all x sufficiently close to x_0, then we say that *f* approaches the **limit** L as x approaches x_0, and we write

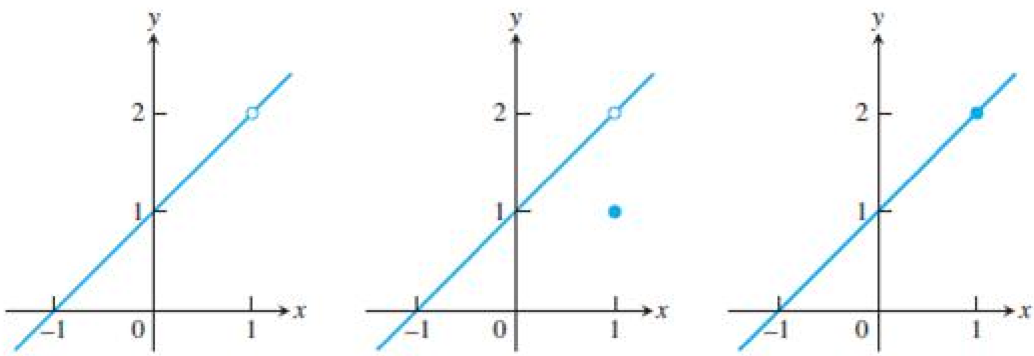
$$\lim_{x \to x_0} f(x) = L$$

The definition says that the values of f(x) are close to L

wherever x close to x_0 on either side of x_0.(either from right or from left of x_0)

This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise. A formal definition will be given later.

The existence of a limit as $x \longrightarrow x_0$ does not depend on how the function may be defined at x_0.



(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

A) The function *f* in the figure above has limit 2 as $x \longrightarrow 1$ even though *f* is not defined at $x = 1$.

B) The function *g* has limit 2 as $x \longrightarrow 1$ even though $2 \neq g(1)$.

C) The function *h* is the only one whose limit as $x \longrightarrow 1$ equals its value at $x = 1$. For h we have $\lim_{x \to 1} h(x) = h(1)$.

Sometimes, $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$.

This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined.

Example:

a) $\lim_{x \to 2}(4) = 4$

b) $\lim_{x \to -13}(4) = 4$

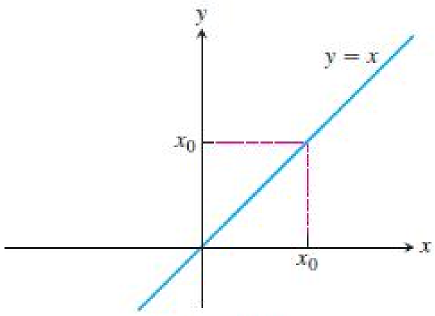
c) $\lim_{x \to 3}(x) = 3$

d) $\lim_{x \to 2}(5x - 3) = 10 - 3 = 7$

e) $\lim_{x \to -2} \frac{3x+4}{x+5} = \frac{-6+4}{-2+5} = -\frac{2}{3}$

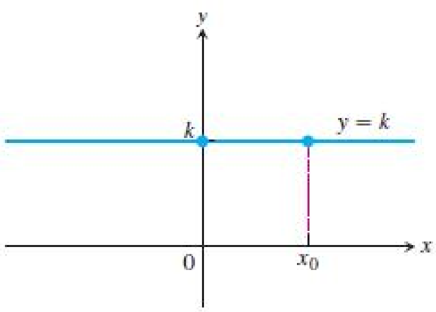
The Identity and Constant Functions Have Limits at Every Point

- If *f* is the **identity function** $f(x) = x$, then for any value of x_0, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$



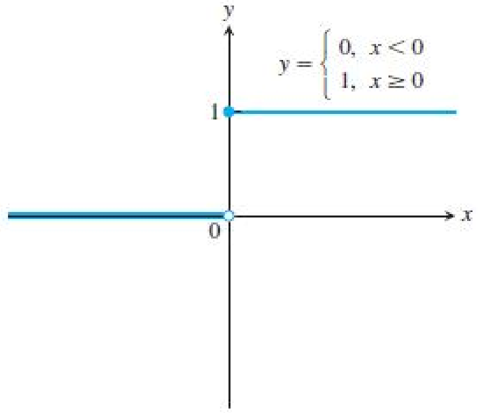
Identity function

- If *f* is the **constant function** $f(x) = k$, then for any value of x_0, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$



Constant function

✓ The function $u(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$ has the following graph



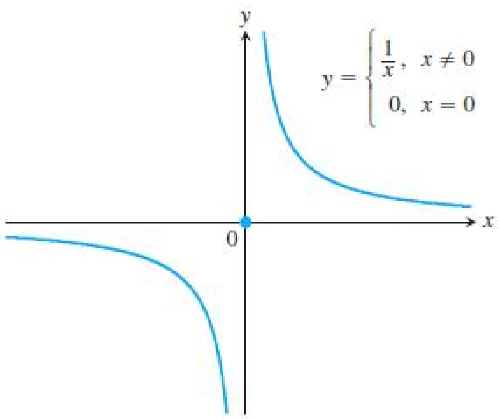
Unit step function $U(x)$

The **unit step function** $u(x)$ has no limit as $x\to 0$

because its values jump at $x = 0$.

for negative values of x arbitrarily close to zero, where $u(x)=0$. For positive values of x arbitrarily close to zero, we have $u(x)=1$. There is no *single* value L approached by $u(x)$ *as* $x\to 0$.

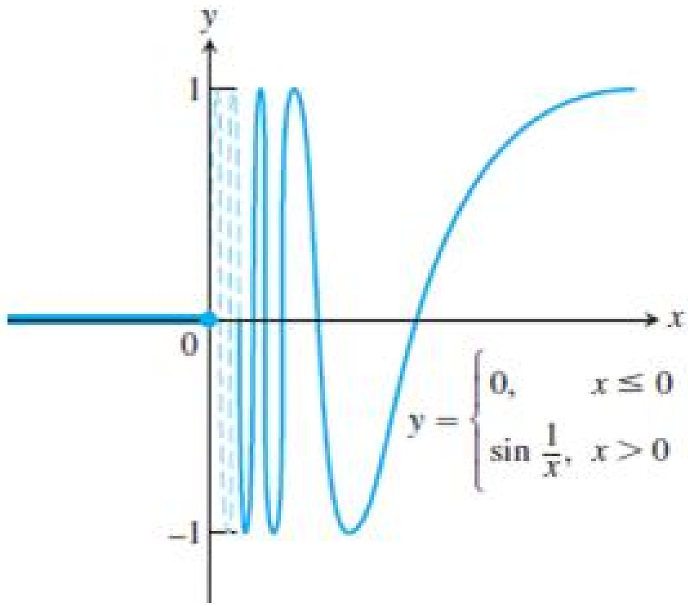
✓ The function $g(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ has the following graph



$g(x)$

$g(x)$ has no limit as $x\to 0$ because the values of g grow ***arbitrarily*** large in absolute value as $x\to 0$ and do not stay close to *any* real number.

✓ The function $f(x)=\begin{cases}0, & x\le 0\\ \sin\dfrac{1}{x}, & x>0\end{cases}$ has the following graph

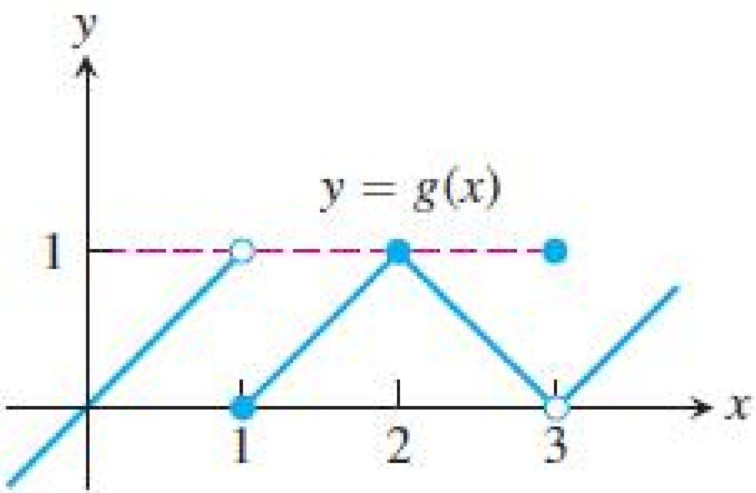


f(x)

f(*x*) has no limit as $x\to 0$ because the function's values oscillate too much between +1 and -1 in every open interval containing 0. The values do not stay close to any one number as $x\to 0$.

P1:

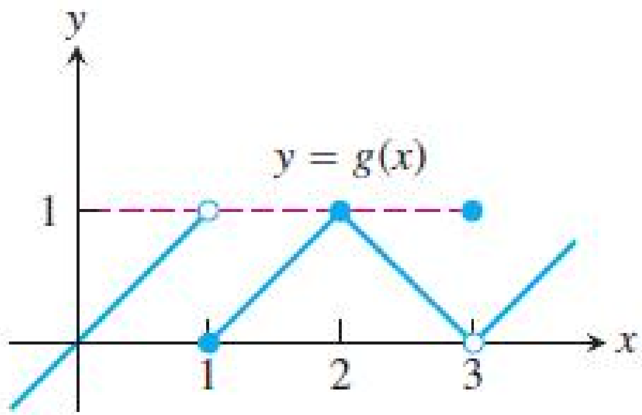
The function g(x) graphed here



Find limit or explain why limit doesn't exist at $\lim_{x \to 2} g(x)$

Solution:

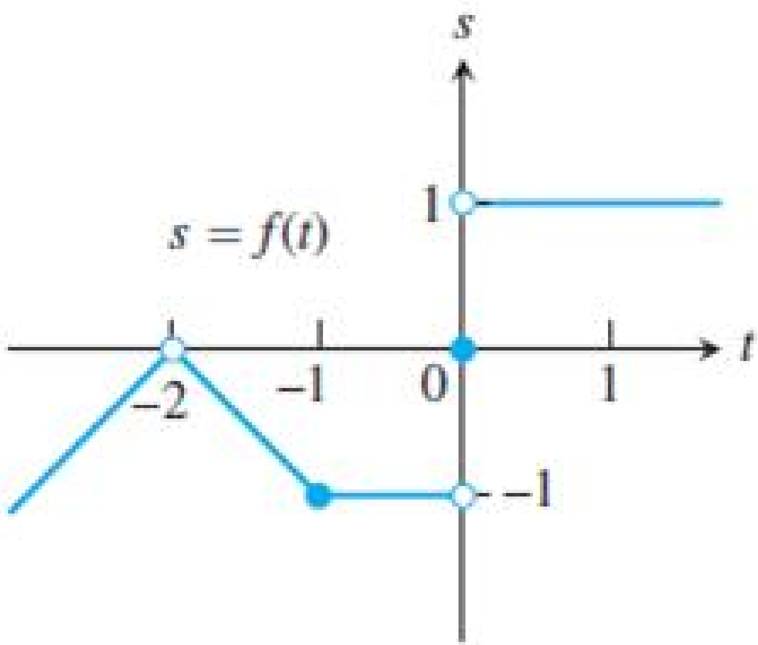
$$\lim_{x \to 2} g(x)$$



From the graph it is clear that g(x) approaches 1 when x approaches 2 from the right,. Also g(x) approaches 1 when x approaches 2 from the left. Thus g(x) approaches 1 when x approaches 2 from either side. Therefore, $\lim_{x \to 2} g(x)$ exists and $\lim_{x \to 2} g(x) = 1.$

P2:

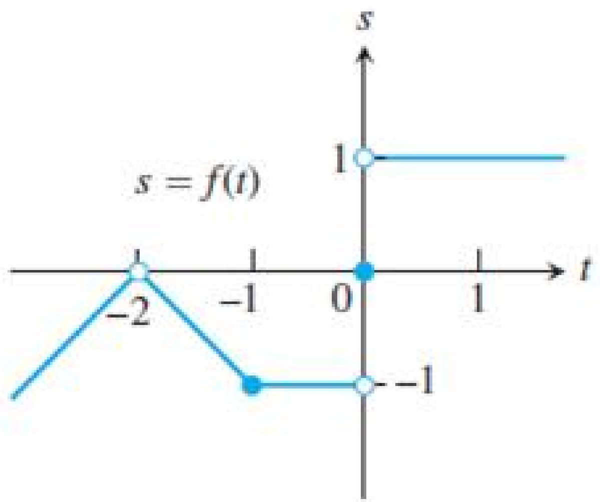
For the function f(t) graphed here.



Find limit or explain why limit doesn't exist for the function $\lim_{t \to 0} f(t)$?

Solution:

$$\lim_{t \to 0} f(t)$$



From the graph it is clear that $f(t)$ approaches 1 when t approaches 0 from the right,. Also $f(t)$ approaches -1 when t approaches 0 from the left. Thus there is no single number L such that $f(t)$ get arbitrarily close to L when t is sufficiently close to 0. There fore,

$\lim_{t \to 0} f(t)$ doesn't exist.

P3:

$$\lim_{x \to -1} 3x(2x-1) =$$

Solution:

Let $f(x) = 3x(2x - 1)$. $f(x)$ is an algebraic polynomial and $f(-1)$ is defined. And

$$\lim_{x \to -1} 3x(2x - 1) = f(-1) = 3(-1)\{2(-1) - 1\} = -3(-3) = 9$$

P4:

$$\lim_{x \to \frac{\pi}{2}} x \, sinx =$$

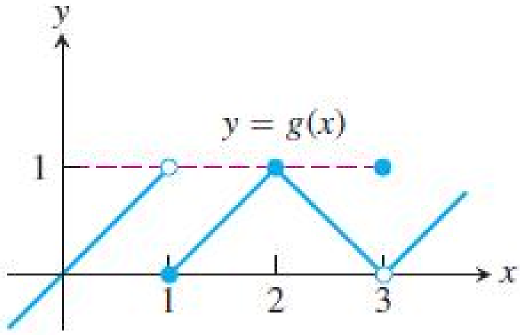
Solution:

Let $f(x) = x\ sinx$. $f(x)$ is a trigonometric function and $f\left(\frac{\pi}{2}\right)$ is defined. And

$$\lim_{x\to\frac{\pi}{2}} x\ sinx = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

IP1:

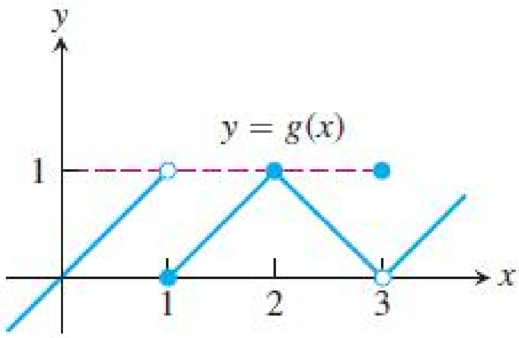
The function g(x) graphed here



Find limit or explain why limit doesn't exist at $\lim_{x \to 1} g(x)$

Solution:

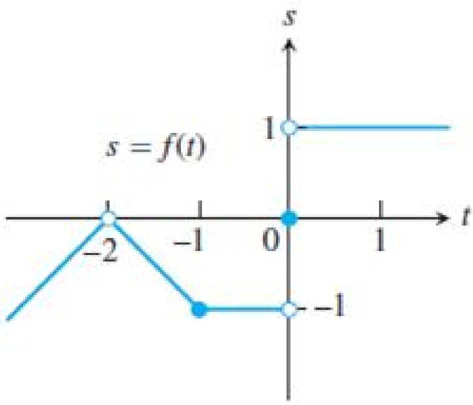
$$\lim_{x \to 1} g(x)$$



From the graph it is clear that g(x) approaches 0 when x approaches 1 from the right,. Also g(x) approaches 1 when x approaches 1 from the left. Thus there is no single number L such that $g(x)$ get arbitrarily close to L when x is sufficiently close to 1. There fore, $\lim_{x \to 1} g(x)$ doesn't exist.

IP2:

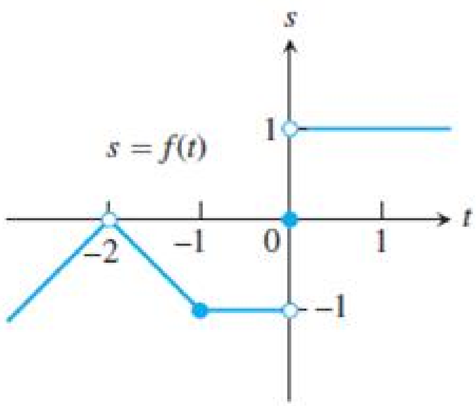
For the function f(t) graphed here.



Find limit or explain why limit doesn't exist for the function $\lim_{t \to -2} f(t)$?

Solution:

$$\lim_{t \to -2} f(t)$$



From the graph it is clear that $f(t)$ approaches 0 when t approaches -2 from the right,. Also $f(t)$ approaches 0 when t approaches -2 from the left. Thus $f(t)$ approaches 0 when t approaches -2 from either side. Therefore, $\lim_{t \to -2} f(t)$ exists and $\lim_{t \to -2} f(t) = 0$.

Notice that $f(-2)$ is not defines.

IP3:

$$\lim_{x \to -1} \frac{3x^2}{2x-1} =$$

Solution:

Let $f(x) = \frac{3x^2}{2x-1}$. $f(x)$ is an algebraic quotient function and $f(-1)$ is defined. And

$$\lim_{x \to -1} \frac{3x^2}{2x-1} = f(-1) = \frac{3(-1)^2}{2(-1)-1} = \frac{3}{2(-1)-1} = \frac{3}{-3} = -1$$

IP4:

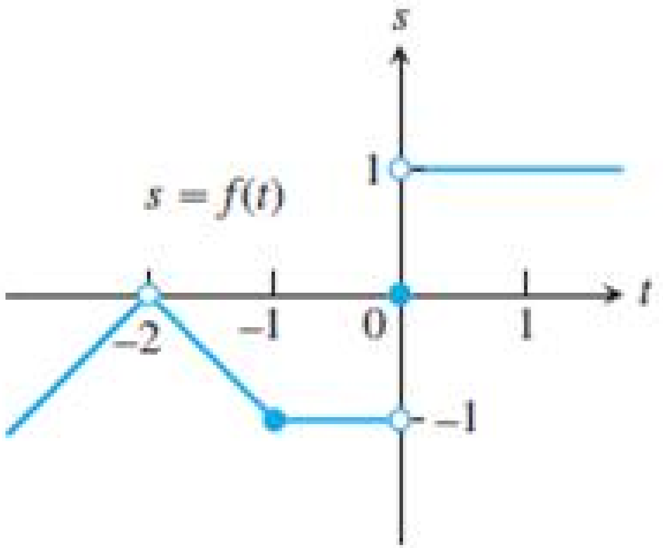
$$\lim_{x\to\pi}\frac{cosx}{1-\pi}=$$

Solution:

Let $f(x)=\frac{cosx}{1-\pi}$. $f(x)$ is a trigonometric function and $f(\pi)$ is defined. And

$$\lim_{x\to\pi}\frac{cosx}{1-\pi}=f(\pi)=\frac{cos\pi}{1-\pi}=\frac{-1}{1-\pi}=\frac{1}{\pi-1}$$

1) For the function **f(t)** graphed here, find the following limits or explain why they do not exist.

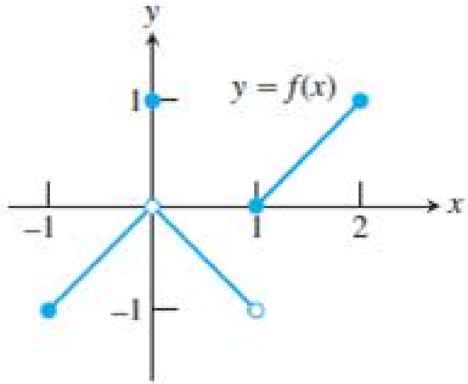


(a) $\lim_{t \to -2} f(t)$

(b) $\lim_{t \to -1} f(t)$

(c) $\lim_{t \to 0} f(t)$

2) . Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



(a) $\lim_{x \to 0} f(x)$ exists

(b) $\lim_{x \to 0} f(x) = 0$

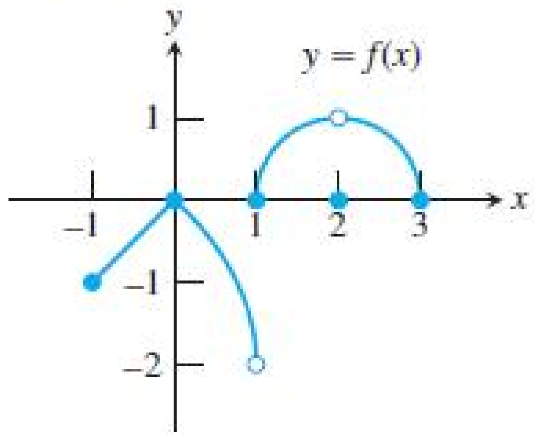
(c) $\lim_{x \to 0} f(x) = 1$

(d) $\lim_{x \to 1} f(x) = 1$

(e) $\lim_{x \to 1} f(x) = 0$

(f) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(-1,1)$

3) Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



(a) $\lim_{x \to 2} f(x)$ does not exist

(b) $\lim_{x \to 2} f(x) = 2$

(c) $\lim_{x \to 1} f(x)$ does not exist

(d) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(-1,1)$

(e) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(1,3)$

4) Find the limits of following functions.

(a) $\lim_{x \to 2} 2x$

(b) $\lim_{x \to 0} 2x$

(c) $\lim_{x \to 1/3} (3x - 1)$

(d) $\lim_{x \to 1} \frac{-1}{3x - 1}$

5.3

Rules for Finding Limits

Learning objectives

i) To state the properties of limits and to apply them to polynomial and rational functions

ii) To state the Sandwich Theorem

And

iii) To find the limits of functions by different techniques

Properties of Limits

Here we state, rules to calculate the limits of functions that are the arithmetic combination of functions whose limits are known.

If L,M,*c* and *k* are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then,

1. Sum Rule: $$\lim_{x \to c} [f(x) + g(x)] = L + M$$

i.e., The limit of the sum of two functions is the sum of their limits

2. Difference Rule: $$\lim_{x \to c} [f(x) - g(x)] = L - M$$

i.e., The limit of the difference of two functions is the difference of their limits.

3. Product Rule: $$\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$$

i.e., The limit of the product of two functions is the product of their limits.

4. Constant Multiple Rule:

$$\lim_{x \to c} kf(x) = kL \qquad (\text{any number } k)$$

i.e., The limit of a constant times a function is that constant times the limit of the function.

5. Quotient Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

i.e., The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. Power Rule: If *m* and *n* are integers, then

$\lim_{x \to c} [f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$, provided $L^{\frac{m}{n}}$ **is a real number.**

i.e., The limit of any rational power of a function is that power of the limit of the function

Example: Find $\lim_{x \to c} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

Solution:

$$\lim_{x \to c} x^3 + 4x^2 - 3 = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

(Sum and difference rule)

$$= \lim_{x \to c} x^3 + 4 \lim_{x \to c} x^2 - 3$$

(Constant multiple rule)

$$= \left(\lim_{x \to c} x\right)^3 + 4\left(\lim_{x \to c} x\right)^2 - 3$$

(Power rule or product rule)

$$= c^3 + 4c^2 - 3 \quad \left(\because \lim_{x \to c} x = c, \lim_{x \to c} k = k\right)$$

$$\lim_{x \to c} x^2 + 5 = \lim_{x \to c} x^2 + \lim_{x \to c} 5 \quad \text{(Sum rule)}$$

$$= \left(\lim_{x \to c} x\right)^2 + \lim_{x \to c} 5 \quad \text{(Power rule)}$$

$$= c^2 + 5 \quad \left(\because \lim_{x \to c} x = c, \lim_{x \to c} k = k\right)$$

Now $$\lim_{x \to c} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{\lim_{x \to c} x^3 + 4x^2 - 3}{\lim_{x \to c} x^2 + 5} \quad \text{(quotient rule)}$$

$$= \frac{c^3 + 4c^2 - 3}{c^2 + 5} \quad \left(\because c^2 + 5 \neq 0\right)$$

Example: Find $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution: $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} \left(4x^2 - 3\right)}$ (Power rule)

$$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} \quad \text{(Difference rule)}$$

$$= \sqrt{4 \lim_{x \to -2} x^2 - 3} \quad \text{(Constant multiple rule)}$$

$$= \sqrt{4\left(\lim_{x \to -2} x\right)^2 - 3} \quad \text{(Power rule)}$$

$$= \sqrt{4(-2)^2 - 3} = \sqrt{13}$$

Limits of Polynomials and Rational Functions

The properties of limits simplify the computation of limits of polynomials and rational functions.

Limits of Polynomials

The limit of a polynomial can be found by substitution. If

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} p(x) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0 = p(c)$$

Limits of Rational Functions

The limit of a rational function can be found by substitution if the limit of the denominator is not zero. If $p(x)$ and $q(x)$ are polynomials and $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

Example:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

The above rule applies only when the denominator of the rational function is not zero at *c*. If the numerator and denominator of a rational function of *x* are both zero at $x = c$, then $(x - c)$ is a common factor. Canceling the common factors in the numerator and denominator will reduce the fraction to one whose denominator is no longer zero at *c*.

When this happens, we can find the limit by substitution in the simplified fraction.

Example: Cancelling a common factor

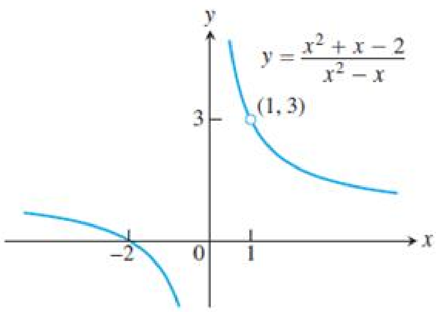
Evaluate $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$

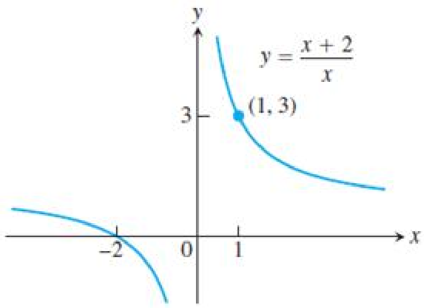
Solution: We cannot just substitute $x = 1$, because it makes the denominator zero. However, we can factor the numerator and denominator and cancel the common factor to obtain

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \quad \text{if } x \neq 1$$

Thus

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$





Example: Creating and cancelling a common factor

Find $\lim\limits_{h \to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$

Solution: We cannot find the limit by substituting $h = 0$, and the numerator and denominator do not have obvious factors. However, we can create a common factor as shown below.

$$\frac{\sqrt{2+h} - \sqrt{2}}{h} = \frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} = \frac{2+h-2}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$

$$= \frac{h}{h\left(\sqrt{2+h} + \sqrt{2}\right)} \quad \text{we have created a common factor of } h$$

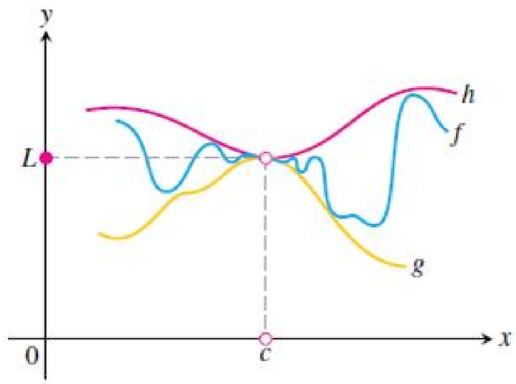
$$= \frac{1}{\sqrt{2+h} + \sqrt{2}}$$

Therefore,

$$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{\sqrt{2+0} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Sandwich Theorem

The Sandwich Theorem refers to a function *f* whose values are sandwiched between the values of two other functions *g* and *h* that have the same limit *L* at a point *c*. Being trapped between the values of two functions that approach *L*, the value of *f* must also approach *L*.



Theorem:

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $$\lim_{x \to c} f(x) = L.$$

Example: Given that $1 - \frac{x^2}{4} \le f(x) \le 1 + \frac{x^2}{2}$ for all $x \ne 0$

Find $\lim_{x \to 0} f(x)$.

Solution: we have,

$$\lim_{x \to 0}\left(1 - \frac{x^2}{4}\right) = 1 \ \textit{and} \ \lim_{x \to 0}\left(1 + \frac{x^2}{2}\right) = 1$$

The Sandwich Theorem implies that $\lim_{x \to 0} f(x) = 1$

Example: Show that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$

Solution:

Since $-|f(x)| \le f(x) \le |f(x)|$, and

$-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches c,

there fore, $\lim_{x \to c} f(x) = 0$ by the Sandwich Theorem.

P1:

$$\lim_{x\to 1} h(x) = 5 \;, \lim_{x\to 1} p(x) = 1 \; and \; \lim_{x\to 1} r(x) = 2 \; then$$

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} =$$

Solution:

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))}=\frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}p(x)(4-r(x))}\text{(Quotient rule)}$$

$$=\frac{\sqrt{\lim_{x\to 1}5h(x)}}{\left(\lim_{x\to 1}p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)}\text{(Difference and product rule)}$$

$$=\frac{\sqrt{5\lim_{x\to 1}h(x)}}{\left(\lim_{x\to 1}p(x)\right)\left(\lim_{x\to 1}4-\lim_{x\to 1}r(x)\right)}\text{(Constant multiple rules)}$$

$$=\frac{\sqrt{5.5}}{(1)(4-2)}=\frac{5}{2}$$

P2:

$$\lim_{x \to \sqrt{2}} \frac{x^3 - 2\sqrt{2}}{x - \sqrt{2}} =$$

Solution:

$$\lim_{x\to\sqrt{2}} \frac{x^3 - 2\sqrt{2}}{x - \sqrt{2}} \Rightarrow \lim_{x\to\sqrt{2}} \frac{x^3 - \left(\sqrt{2}\right)^3}{x - \sqrt{2}}$$

$$\Rightarrow \lim_{x\to\sqrt{2}} \frac{\left(x - \sqrt{2}\right)\left(x^2 + \sqrt{2}x + 2\right)}{x - \sqrt{2}} \Rightarrow \lim_{x\to\sqrt{2}} \left(x^2 + \sqrt{2}x + 2\right)$$

$$\lim_{x\to\sqrt{2}} \left(x^2 + \sqrt{2}x + 2\right) = 2 + 2 + 2 = 6$$

$$\lim_{x\to\sqrt{2}} \frac{x^3 - 2\sqrt{2}}{x - \sqrt{2}} = 6$$

P3:

If $\lim_{x \to 4} \frac{f(x)-5}{x-4} = 1$, find $\lim_{x \to 4} f(x) =$

Solution:

If $\lim_{x \to 4} \frac{f(x)-5}{x-4} = 1$, find $\lim_{x \to 4} f(x) =$

Solution:

$$\lim_{x \to 4} \frac{f(x)-5}{x-4} = 1$$

$$\Rightarrow \lim_{x \to 4} f(x) - 5 = \lim_{x \to 4} \frac{f(x)-5}{(x-4)}(x-4)$$

$$= \lim_{x \to 4} \frac{f(x)-5}{(x-4)} \times \lim_{x \to 4}(x-4)$$

$$= 1 \times 0 = 0$$

$$\Rightarrow \lim_{x \to 4} f(x) - \lim_{x \to 4} 5 = 0$$

$$\Rightarrow \lim_{x \to 4} f(x) = 5$$

P4:

If $4 - x^2 \leq g(x) \leq 4cosx$ for all x, find $\lim_{x \to 0} g(x) =$

Solution:

We have $4 - x^2 \leq g(x) \leq 4cosx \ \forall \ x \in R$, and

$\lim_{x\to 0} 4 - x^2 = 4 - 0 = 4$, $\lim_{x\to 0} 4\, cosx = 4.1 = 4$

By Sandwich theorem $\lim_{x \to 0} g(x) = 4$

IP1:

Suppose $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to 0} g(x) = 5$. Then

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{\frac{2}{3}}} =$$

Solution:

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{\frac{2}{3}}} = \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\lim_{x\to 0}(f(x)+7)^{\frac{2}{3}}}$$ (Quotient rule)

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{\frac{2}{3}}}$$ (Power and product rule)

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{\frac{2}{3}}}$$ (difference and constant

multiples rule)

$$= \frac{2(1)-5}{(1+7)^{\frac{2}{3}}} = \frac{-3}{(1+7)^{\frac{2}{3}}} = \frac{-3}{4}$$

IP2:

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}=$$

Solution:

Step1:

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}=\lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}$$

Step2:

$$=\lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x}$$

Step3:

$$=\lim_{x\to 0}\left(\sqrt{x+4}+2\right)=4$$

IP3:

If $\lim_{x \to 2} \frac{f(x)-3}{x-2} = 3$, find $\lim_{x \to 2} f(x) =$

Solution:

$$\lim_{x \to 2} \frac{f(x)-3}{x-2} = 3$$

$$\Rightarrow \lim_{x \to 2} (f(x) - 3) = \lim_{x \to 2} \frac{f(x)-3}{(x-2)}(x-2)$$

$$= \lim_{x \to 2} \frac{f(x)-3}{(x-2)} \times \lim_{x \to 2} (x-2)$$

$$= 3 \times 0 = 0$$

$$\Rightarrow \lim_{x \to 2} f(x) - \lim_{x \to 2} 3 = 0$$

$$\Rightarrow \lim_{x \to 2} f(x) = 3$$

IP4:

If $\sqrt{9 - 4x^2} \leq g(x) \leq \sqrt{9 - 3x^2}$ for all $-1 \leq x \leq 1$, find $\lim\limits_{x \to 0} g(x) =$

Solution:

We have $\sqrt{9 - 4x^2} \leq g(x) \leq \sqrt{9 - 3x^2}$ for all $-1 \leq x \leq 1$

and,

$\lim\limits_{x \to 0} \sqrt{9 - 4x^2} = \sqrt{9} = 3$, $\lim\limits_{x \to 0} \sqrt{9 - 3x^2} = \sqrt{9} = 3$

By Sandwich theorem $\lim\limits_{x \to 0} g(x) = 3$

1. Evaluate

a) $\lim_{x \to -2} \left(x^3 - 2x^2 + 4x + 8 \right)$

b) $\lim_{h \to 0} \frac{3}{\sqrt{3h+1}+1}$

c) $\lim_{t \to 2} \frac{t+3}{t+6}$

d) $\lim_{t \to c} f(t) = 5, \quad \lim_{t \to c} g(t) = -2 \; find \; \lim_{t \to c} \frac{f(t)}{f(t) - g(t)} =$

2. Evaluate

a) $\lim_{x \to 5} \frac{x-5}{x^2-25}$

b) $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$

c) $\lim_{x \to 2} \frac{x^2-7x+10}{x-2}$

d) $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

3. Evaluate

a) $\lim_{h \to 0} \frac{\sqrt{3h+1}-1}{h}$

b) $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

c) $\lim_{x \to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

d) $\lim_{x \to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

4. Evaluate $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for the following cases

a. $f(x) = x^2$, $\quad x = 1$

b. $f(x) = \dfrac{1}{x}$ $\quad x = -2$

c. $f(x) = \sqrt{x}$ $\quad x = 7$

d. $f(x) = \sqrt{3x+1}$ $\quad x = 0$

5.

a) If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim\limits_{x \to 0} f(x)$

b) Find $\lim\limits_{x \to 0} \dfrac{x \sin x}{2 - \cos 2x}$ given that the inequalities $1 - \dfrac{x^2}{6} \le \dfrac{x \sin 2x}{2 - 2\cos 2x} \le 1$ hold for all values of x close to zero.

c) Find $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ given that the inequalities $\dfrac{1}{2} - \dfrac{x^2}{24} \le \dfrac{1 - \cos x}{x^2} \le \dfrac{1}{2}$ hold for all values of x close to zero.

6) Evaluate

a) If $\lim\limits_{x \to 4} \dfrac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x \to 4} f(x)$

b) If $\lim\limits_{x \to 2} \dfrac{f(x)-5}{x-2} = 3$, find $\lim\limits_{x \to 2} f(x)$

c) If $\lim\limits_{x \to 2} \dfrac{f(x)-5}{x-2} = 4$, find $\lim\limits_{x \to 2} f(x)$

d) If $\lim\limits_{x \to 1} \dfrac{f(x)-8}{x-1} = 10$, find $\lim\limits_{x \to 1} f(x)$

5.5

Extensions of the Limit Concept

Learning objectives:

- To define right and left hand limits
- To define the limit in terms of one sided limits

  And

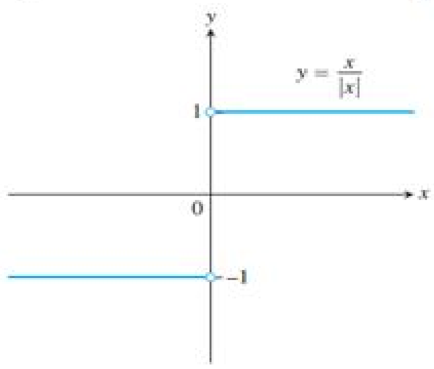
- To practice related problems.

Now we extend the concept of limit to ***one-sided limits***, which are limits as x approaches a from the left-hand side(where $x < a$) or the right-hand side (where $x > a$) only.

One Sided Limits

To have a limit *L* as x approaches *a*, a function *f* must be defined on both sides of *a*, and its value *f*(x) must approach *L* as x approaches *a* from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches *a* from only one side, either from the right or from the left. In this case we say that *f* has a **one-sided** limit at *a*. The function $f(x) = \frac{x}{|x|}$ graphed below has limit 1 as *x* approaches zero from the right, and limit −1 as *x* approaches zero from the left.



Definition

Let $f(x)$ defined on an interval (a,b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x \to a^+} f(x) = L$$

Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within that interval, then we say that f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M$$

For the function $f(x) = \frac{x}{|x|}$ in the figure above, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.

Definition

Let *f*(x) defined on an interval (*a*,*b*) where *a* < *b*. If *f*(x) approaches arbitrarily close to *L* as x approaches a from within that interval, then we say that *f* has **right-hand limit** *L* at *a*, and we write

$$\lim_{x \to a^+} f(x) = L$$

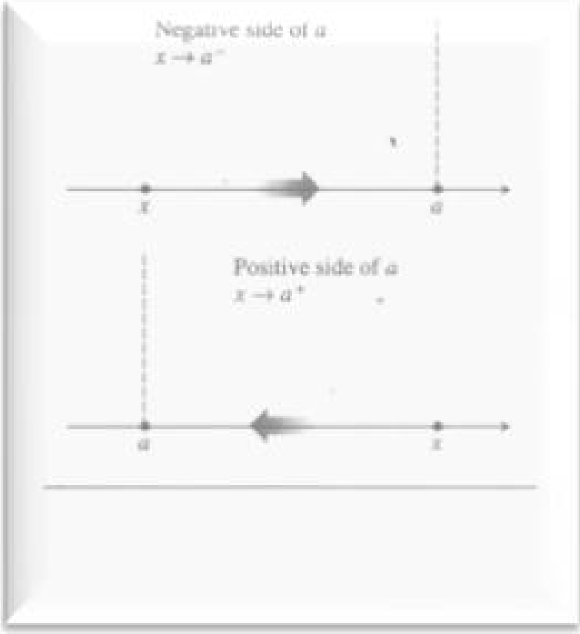
Let *f*(*x*) be defined on an interval (*c*, *a*) where *c* < *a*. If *f*(*x*) approaches arbitrarily close to *M* as *x* approaches *a* from within that interval, then we say that *f* has **left-hand limit** *M* at *a*, and we write

$$\lim_{x \to a^-} f(x) = M$$

For the function $f(x) = \frac{x}{|x|}$ in the figure above, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

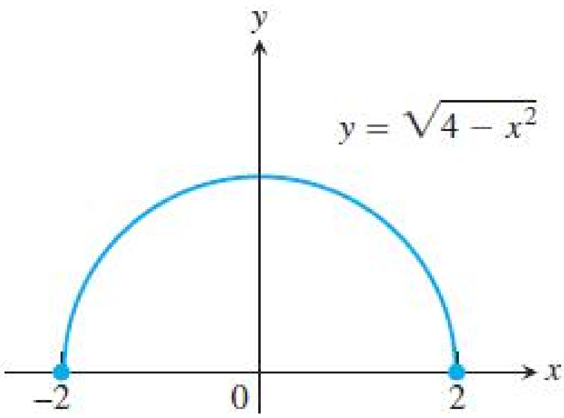
A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.



The symbol $x \to a^-$ means x approaches a from the negative side of a, through values less than a.

The symbol $x \to a^+$ means x approaches a from the positive side of a, through values greater than a.

Example: The domain of $f(x) = \sqrt{4 - x^2}$ *is* $[-2,2]$; its graph is semi-circle shown below.



$$\lim_{x \to -2^+} f(x) = 0 \text{ , } \lim_{x \to 2^-} f(x) = 0$$

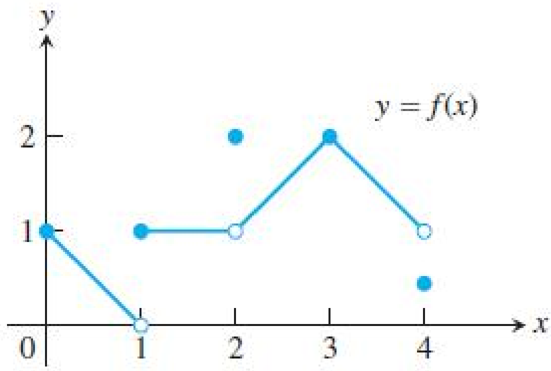
The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2.

One-sided versus two-sided Limits

A function $f(x)$ has a limit as *x* approaches *c* if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L$$

Example: All of the following statements about the function graphed in the figure below are true.



At $x = 0$

$\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of x = 0.

At $x = 1$

$\lim_{x\to 1^+} f(x) = 0$ even though f(1) = 1

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x)$ does not exist. The right-hand and left-hand limits are not equal.

At $x = 2$

$\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2} f(x) = 1$ even though *f*(2) = 2.

At $x = 3$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$

At $x = 4$

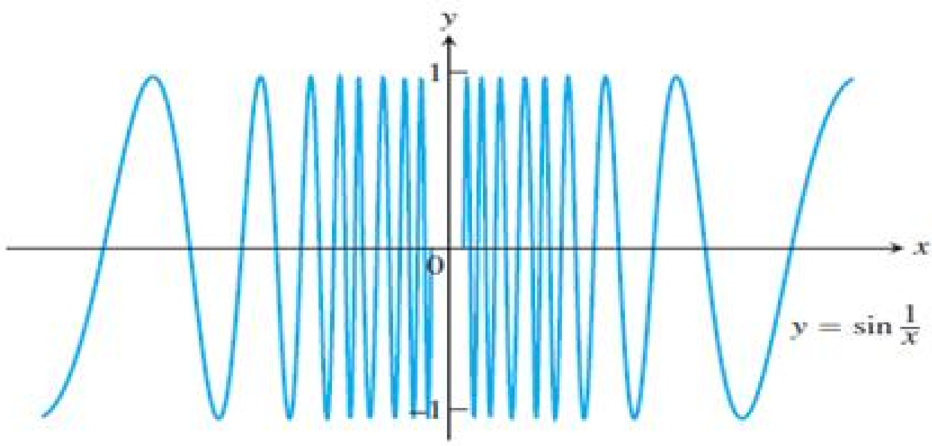
$\lim_{x \to 4^-} f(x) = 1$ even though f(4) ≠ 1.

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point *a* in [0,4], *f*(*x*) has limit *f*(*a*).

The function $y = \sin\left(\frac{1}{x}\right)$ has neither a right-hand nor a left-hand limit as *x* approaches zero. This can be seen from the following observations.

As x approaches zero, its reciprocal $\frac{1}{x}$ grows without bound and the value of $\sin(1/x)$ cycle repeatedly from −1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive or negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.



The formal definition of two-sided limits can be easily modified for one-sided limits.

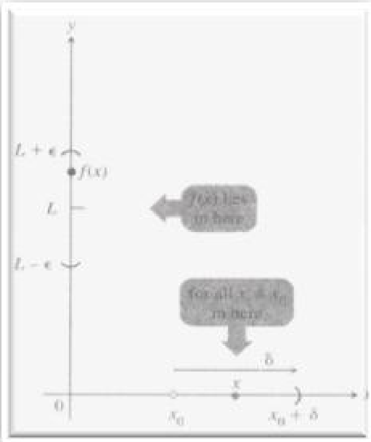
Right-hand Limit

We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \quad |f(x) - L| < \varepsilon$$



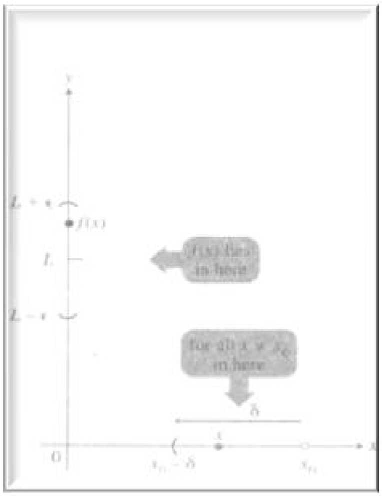
Left-hand Limit

We say that $f(x)$ has left-hand limit L at x_0, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \quad |f(x) - L| < \varepsilon$$



P1)

Evaluate the left hand limit of the function

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad at\ x = 4.$$

Solution:

Given, $f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases}$ $at\ x = 4$

If $x < 4$ then $x - 4 < 0$ and $|x - 4| = -(x - 4)$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{|x - 4|}{x - 4} = \lim_{x \to 4^-} \frac{-(x - 4)}{x - 4} = -1$$

P2)

$$f(x) = \begin{cases} 1 + x^2, & if \quad 0 \leq x \leq 1 \\ 2 - x \;, & if \quad x > 1 \end{cases} \qquad at \;\; x = 1.$$

show that $\lim_{x \to 1} f(x)$ does not exist.

Solution:

We have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 + x^2)$$

$$= \lim_{h \to 0} (1 + (1 - h)^2) = 1 + 1 = 2$$

and,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x)$$

$$= \lim_{h \to 0} (2 - (1 + h)) = 1$$

Thus $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Therefore, $\lim_{x \to 1} f(x)$ does not exist.

P3)

$f(x) = \begin{cases} cosx, & if \quad x \geq 0 \\ x + k, & if \quad x < 0 \end{cases}$. Find the value of constant k, given that $\lim_{x \to 0} f(x)$ exists.

Solution:

We have,

$$\lim_{x \to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^-}(x + k) = \lim_{x\to 0^+} cosx$$

$$\Rightarrow \lim_{h\to 0}(0 - h + k) = \lim_{h\to 0} \cos(0 + h)$$

$$\Rightarrow \mathrm{k} = \lim_{h\to 0} \cos(h) \Rightarrow \mathrm{k} = 1$$

P4)

If $f: R \to \mathbb{R}$ is an odd function and if $\mathbf{lim}_{x \to 0} f(x)$ exists. Prove that this limit must be zero.

Solution:

It is given that, $f(-x) = -f(x), \forall\ x$

$\lim_{x\to 0} f(x)$ exists

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \quad \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$

$\Rightarrow \quad \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h)$

$\Rightarrow \quad \lim_{h\to 0}(-f(h)) = \lim_{x\to 0} f(h)$

[$\because f(x)$ is odd]

$\Rightarrow \quad -\lim\limits_{h\to 0} f(h) = \lim\limits_{x\to 0} f(h)$

$\Rightarrow \quad 2\ \lim_{h\to 0} = 0 \Rightarrow \lim_{h\to 0} f(h) = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$

IP1)

Evaluate the right hand limit of the function

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad at \ x = 4.$$

Solution:

Given, $f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad at \ x = 4$

If $x > 4$ then $x - 4 > 0$ and $|x - 4| = (x - 4)$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{|x-4|}{x-4} = \lim_{x \to 4^+} \frac{(x-4)}{x-4} = 1$$

IP2)

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$$

Solution:

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x\to 1^+} \sqrt{2x}\;.\lim_{x\to 1^+} \frac{(x-1)}{|x-1|} \quad \text{(Product rule)}$$

$$= \lim_{x\to 1^+} \sqrt{2x}\;.1 \quad \left(\text{Since } \lim_{x\to 1^+} \frac{(x-1)}{|x-1|} = 1\right)$$

$$= \lim_{h\to 0} \sqrt{2(1+\mathrm{h})}$$

$$= \sqrt{\lim_{h\to 0} 2(1+\mathrm{h})} \quad \text{(Power rule)}$$

$$= \sqrt{2}$$

IP3)

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$$

For what values of integers m, n does the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

Solution:

1)

$\lim_{x\to 0} f(x)$ exist

$\Rightarrow$ $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ both exist. And they are equal. Now ,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \Rightarrow \lim_{x\to 0^+}(nx + m) = \lim_{x\to 0^-}(mx^2 + n)$$

$$\Rightarrow \lim_{h\to 0}(n(0 - h) + m) = \lim_{h\to 0}(m(0 + h)^2 + n)$$

$$\Rightarrow \lim_{h\to 0}(-nh + m) = \lim_{h\to 0}(mh^2 + n) \Rightarrow m = n$$

Therefore, $\lim_{x\to 0} f(x)$ exists for all values of m, n *such that* $m = n$.

2)

$\lim_{x\to 1} f(x)$ exist

$\Rightarrow$ $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$ both exist, and they are equal.

Now , $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$$\Rightarrow \lim_{x\to 1^-}(nx + m) = \lim_{x\to 1^+}(nx^3 + m)$$

$$\Rightarrow \lim_{h\to 0}(n(1 - h) + m) = \lim_{h\to 0}(n(1 + h)^3 + m)$$

$$\Rightarrow n + m = n + m$$

Therefore, $\lim_{x\to 1} f(x)$ exists for all values of m, n .

Now the values of m, n for which the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists is $m = n$

IP4)

If $f:\mathbb{R}\to\mathbb{R}$ is an even function, then prove that $\lim_{x\to 0} f(x)$ exists.

Solution:

We have, $f(-x)=f(x)$, $\forall\, x$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

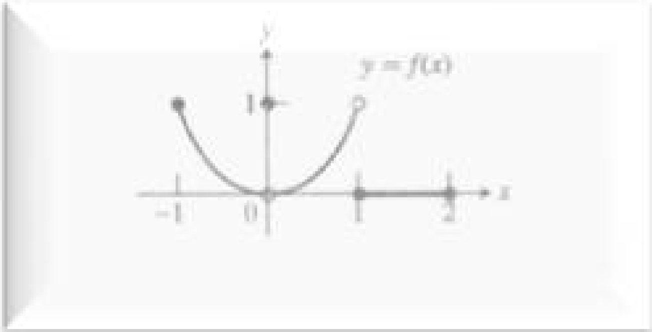
$$= \lim_{h\to 0} f(h) \qquad [\because f(x) \text{ is even }]$$

$$= \lim_{h\to 0} f(0+h) = \lim_{x\to 0^+} f(x).$$

Therefore $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

Then f is an even function

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to 1^+} f(x) = 1$

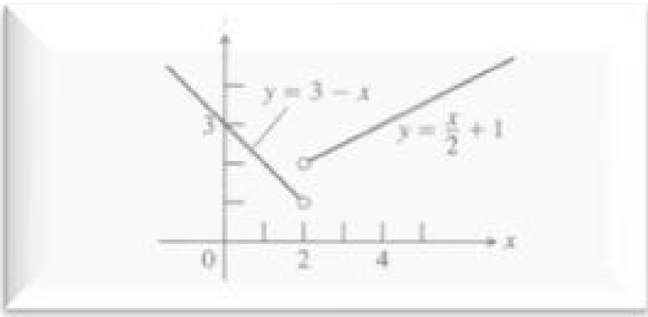
b. $\lim_{x \to 0^-} f(x) = 0$ $\lim_{x \to 0^-} f(x) = 1$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

c. $\lim_{x \to 0} f(x)$ exists $\lim_{x \to 0} f(x) = 0$ $\lim_{x \to 0} f(x) = 1$

d. $\lim_{x \to 1} f(x) = 1$ $\lim_{x \to 1} f(x) = 0$

e. $\lim_{x \to 2^-} f(x) = 2$ $\lim_{x \to -1^-} f(x)$ does not exist $\lim_{x \to 2^+} f(x) = 0$

2. Let $f(x)=\begin{cases} 3-x & x<2 \\ \dfrac{x}{2}+1 & x>2 \end{cases}$



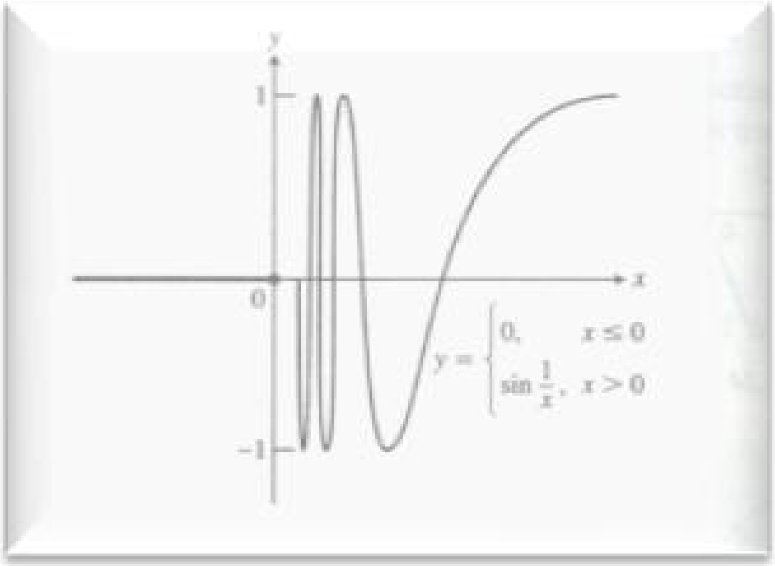
a. Find $\lim\limits_{x\to 2^+} f(x)$ and $\lim\limits_{x\to 2^-} f(x)$

b. Does $\lim\limits_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim\limits_{x\to 4^-} f(x)$ and $\lim\limits_{x\to 4^+} f(x)$

d. Does $\lim\limits_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

3. Let $f(x)=\begin{cases} 0 & x \le 0 \\ \sin\dfrac{1}{x} & x > 0 \end{cases}$



a. Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

4.

a. Graph $f(x)=\begin{cases} x^3 & x\neq 1 \\ 0 & x=1 \end{cases}$

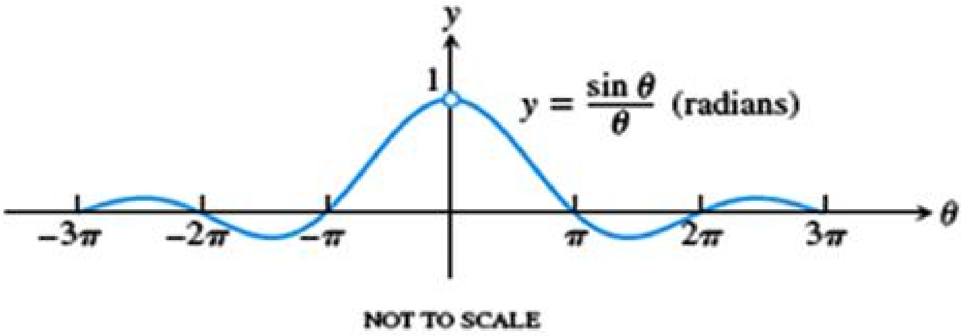
b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Limits involving $\frac{\sin\theta}{\theta}$

We have already noted that $\lim_{\theta\to 0} sin\,\theta = 0$ and $\lim_{\theta\to 0} cos\,\theta = 1$.

We now, take up $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$, where θ is measured in Radian measure. It may be seen $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ from the following figure



The graph of $f(\theta) = \frac{(\sin\theta)}{\theta}$

Notice that $\sin\theta$ and θ are odd functions.Therefore

$f(\theta) = \frac{\sin\theta}{\theta}$ is an even function with a graph symmetry about the y −axis (see the above fig). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$

Therefore, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

We prove the above result algebraically in a subsequent module.

Example:

Find (i) $\lim_{x\to 0}\frac{cosx-1}{x}$ (ii) $\lim_{x\to 0}\frac{\sin 3x}{2x}$

Solution:

$$(i)\ \lim_{x\to 0}\frac{cosx-1}{x}=\lim_{x\to 0}\frac{-2\ sin^2\frac{x}{2}}{x}=-\lim_{x\to 0}\frac{sin\frac{x}{2}}{\left(\frac{x}{2}\right)}.\lim_{x\to 0} sin\frac{x}{2}$$

$$=-1.0=0$$

$$(ii)\ \lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\left(\frac{3}{2}\right)\frac{\sin 3x}{3x}$$

$$=\frac{3}{2}.\lim_{x\to 0}\frac{\sin 3x}{3x}$$

Put $\theta=3x$. Now, $\theta\to 0$ as $x\to 0$.

$$=\left(\frac{3}{2}\right).\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{3}{2}\ .1=\frac{3}{2}$$

Finite limits as $x \longrightarrow \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range out grow all finite bounds.

For example the function $f(x) = \frac{1}{x}$ is defined for all $x \in \mathbb{R}$, $x \neq 0$. If x is positive and becomes increasingly large then $\frac{1}{x}$ becomes increasingly small. Further, If x is negative and its magnitude becomes increasingly large then $\frac{1}{x}$ again becomes small. Summarizing these observations, we say that $f(x) = \frac{1}{x}$ has limit 0 as $x \longrightarrow \pm\infty$(or that 0 is the limit of $f(x) = \frac{1}{x}$ at infinity and negative infinity.)

Definition: Limits as $x \longrightarrow \infty$ or $-\infty$

1. We say that $f(x)$ **has the limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if for every number $\in > 0$, there exists a corresponding number M such that for all x

$$x > M \Rightarrow |f(x) - L| < \in.$$

2. We say that $f(x)$ **has the limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if for every number $\in > 0$, there exists a corresponding number N such that,

$$x < N \Rightarrow |f(x) - L| < \in.$$

That is,

(1) $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L whenever x moves increasingly far from the origin in the positive direction.

(2) $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L whenever x moves increasingly far from the origin in the negative direction.

The calculation of limits of functions as $x \to \pm\infty$ is similar to the one for finite limits, as discussed in the earlier modules. Further the limits at infinity have properties similar to those of finite limits. We have,

$$\lim_{x\to\pm\infty} k = k \text{ and } \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

Example:

a) Find $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$

Solution:

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \quad \text{(sum rule)}$$

$$= 5 + 0 = 5$$

b) Find $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2}$

Solution:

$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \pi\sqrt{3}.\lim_{x\to-\infty}\frac{1}{x^2}$$

$$= \pi\sqrt{3}.\left(\lim_{x\to-\infty}\frac{1}{x}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right) \quad \text{(Product rule)}$$

$$= 0$$

Limits at infinity of rational function

To determine the limit of a rational function as $x \to \pm\infty$, we divide the numerator and denominator by the highest power of x in the denominator and apply the limits.

Numerator and denominator of same degree

Example:

Find $\lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2}$

Solution:

$$\lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x \to \infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}$$

(by dividing numerator and denominator by the highest power of x in the denominator ,i.e., x^2).

$$= \frac{\lim_{x\to\infty}\left(5+\frac{8}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{2}{x^2}\right)} = \frac{5}{3}$$

Degree of Numerator less than degree of denominator.

Example:

Find $\lim_{x \to -\infty} \frac{11x+2}{2x^3-1}$

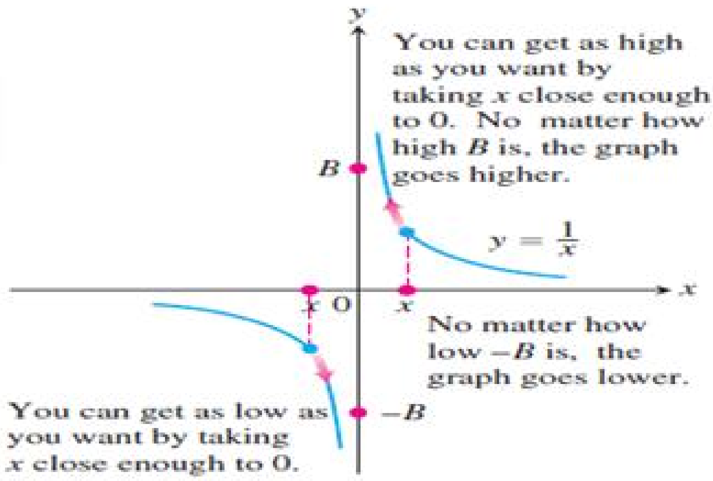
Solution:

$$\lim_{x \to -\infty} \frac{11x+2}{2x^3-1} = \lim_{x \to -\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$$

(by dividing numerator and denominator by the highest power of x in the denominator i.e., x^3).

$$= \frac{\lim_{x \to -\infty}\left(\frac{11}{x^2}+\frac{2}{x^3}\right)}{\lim_{x \to -\infty}\left(2-\frac{1}{x^3}\right)} = \frac{0+0}{2-0} = 0.$$

Consider the function $\mathbf{f(x)} = \frac{1}{\mathbf{x}}$. Its graph is shown below.



As $\mathbf{x} \to 0^+$, the values of *f* grow without bound. That is, given any positive real number *B*, however large, the values of *f* become larger still. Thus *f has no limit as* $\mathbf{x} \to 0^+$. However, *it is conventional to say that* $\mathbf{f(x)}$ *approaches* ∞ *(although there is no such number* ∞*) as* $\mathbf{x} \to 0^+$. *We write*

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

As $x \longrightarrow 0^-$, the values of $\mathbf{f(x)} = \frac{1}{\mathbf{x}}$ become arbitrarily large and negative. Given any negative real number –B, the values of *f* eventually lie below –B. Thus *f has no limit as* $x \longrightarrow 0^-$. We write

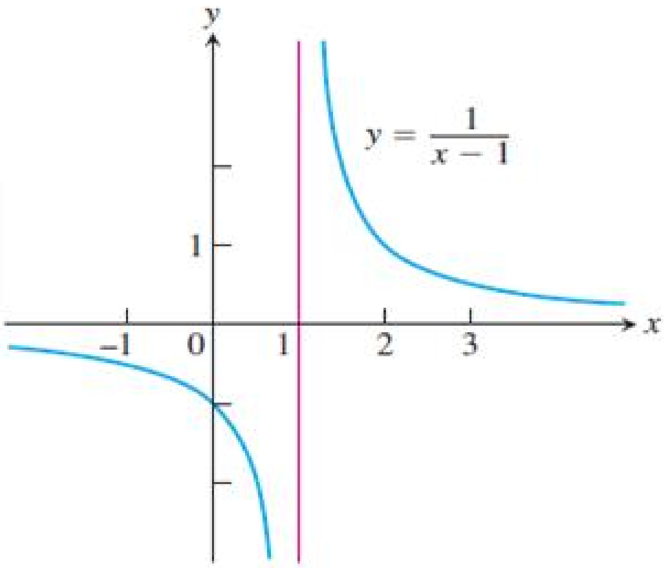
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The notion of infinite limit and the symbol ∞ facilitate the description of the behaviour of functions whose values become arbitrarily large.

One - sided infinite limits

Example: Find $\lim_{x \to 1^+} \frac{1}{x-1}$ and $\lim_{x \to 1^-} \frac{1}{x-1}$

Solution: The graph of $\mathbf{y} = \frac{1}{\mathbf{x}-1}$ is the graph of of $\mathbf{y} = \frac{1}{\mathbf{x}}$ shifted 1 unit to the right. Therefore, the behaviour of $\frac{1}{x-1}$ near $x = 1$ is exactly same as the behaviour of $\frac{1}{x}$ at $x = 0$.



Therefore, $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$ and $\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$

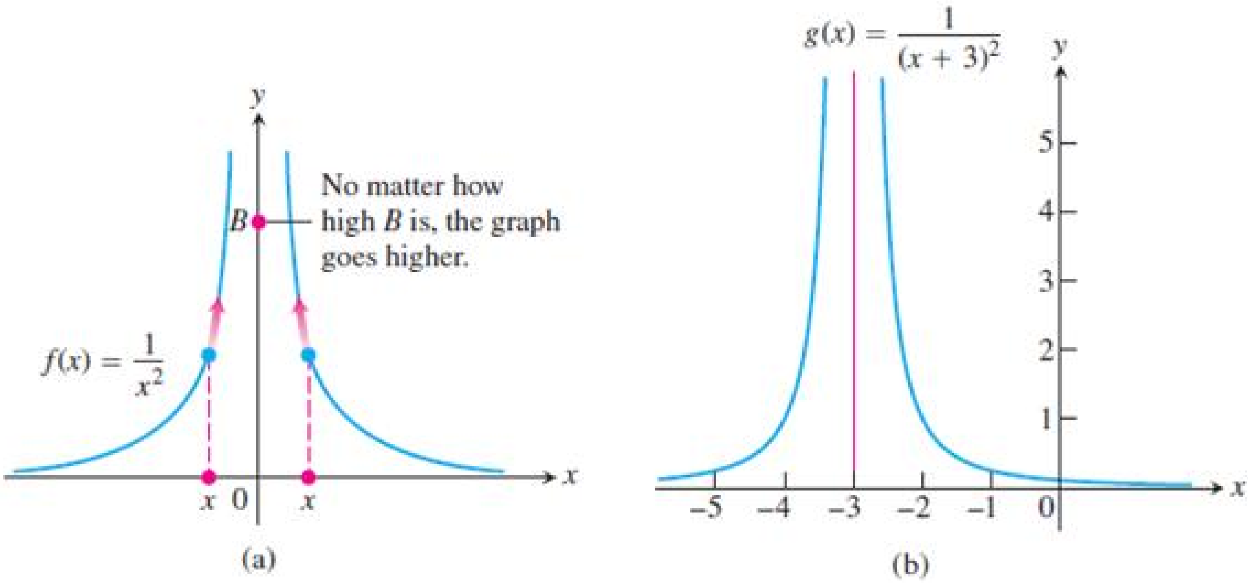
We can also think: As $\mathbf{x} \to 1^+$, we have $(\mathbf{x}-1) \to 0^+$ and $\frac{1}{\mathbf{x}-1} \to \infty$. As $\mathbf{x} \to 1^-$, we have $(\mathbf{x}-1) \to 0^-$ and $\frac{1}{\mathbf{x}-1} \to -\infty$.

Two sided infinite limits

Example: Discuss the behaviour of

a) $f(x) = \frac{1}{x^2}$, near $x = 0$,

b) $g(x) = \frac{1}{(x+3)^2}$, near $x = -3$



Solution:

a) As $x \longrightarrow 0$ *from either side*, the values of $\frac{1}{x^2}$ are positive and becomes arbitrarily large. Therefore,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

b) The graph of $g(x) = \frac{1}{(x+3)^2}$ is the graph of $f(x) = \frac{1}{x^2}$ shifting 3 units to the left. Therefore the behaviour of g near $x = -3$ is exactly same as $f(x) = \frac{1}{x^2}$ near x=0.

Therefore, $\lim_{x \to -3} g(x) = \lim_{x \to -3} \frac{1}{(x+3)^2} = \infty$

Note:

1) The function $y = \frac{1}{x}$ shows no consistent behaviour as $x \longrightarrow 0$. We have seen that $\frac{1}{x} \longrightarrow \infty$ as $x \longrightarrow 0^+$ and $\frac{1}{x} \longrightarrow -\infty$ as $x \longrightarrow 0^-$. In this sense we say that $\lim_{x \to 0} f(x)$ doesn't exist.

2) The values of the function $y = \frac{1}{x^2}$ approaches ∞ as $x \longrightarrow 0$ from either side. In this sense we say that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Rational functions can behave in various ways near zeros of their denominators.

Example: We consider the limits of certain rational functions near zeros of their denominators.

a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

b) $\lim_{x\to 2} \frac{(x-2)}{x^2-4} = \lim_{x\to 2} \frac{(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

c) $\lim_{x\to 2^+} \frac{(x-3)}{x^2-4} = \lim_{x\to 2^+} \frac{(x-3)}{(x-2)(x+2)} = -\infty$

(The values are –ve for x>2, x close to 2)

d) $\lim_{x\to 2^-} \frac{(x-3)}{x^2-4} = \lim_{x\to 2^-} \frac{(x-3)}{(x-2)(x+2)} = \infty$

(The values are +ve for x<2, x close to 2)

e) $\lim_{x\to 2} \frac{(x-3)}{x^2-4} = \lim_{x\to 2} \frac{(x-3)}{(x-2)(x+2)}$ does not exist(by c and d)

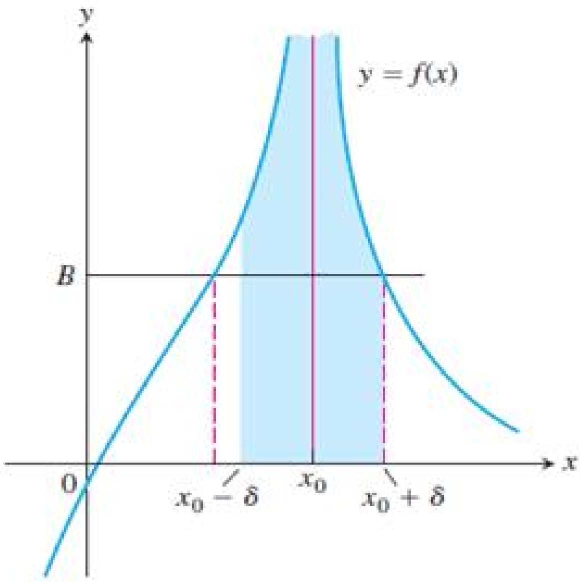
f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

Precise definitions of infinite limits can also be formulated in the same way, as we did in the previous module. *Instead of requiring* **f(x)** *to lie arbitrarily close to a finite number* **L** *for all* **x** *sufficiently close to* x_0, *the definitions of infinite limits require* **f(x)** *to lie arbitrarily far from the origin.*

Infinite Limits

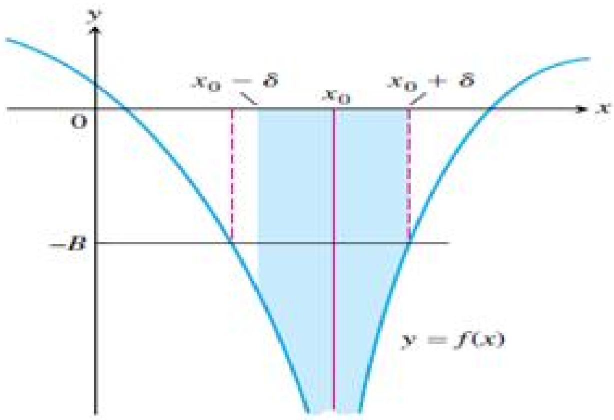
We say that **f(x)** *approaches infinity* as **x** *approaches* $\mathbf{x}_0$, and write $\lim\limits_{x \to x_0} f(x) = \infty$, if for every positive real number *B* there exists a corresponding number $\delta > 0$ such that for all **x**

$$0 < |x - x_0| < \delta \;\Rightarrow\; f(x) > B$$



We say that **f(x)** *approaches minus infinity* as **x** *approaches* $\mathbf{x}_0$, and write $\lim\limits_{x \to x_0} f(x) = -\infty$, if for every negative real number −*B* there exists a corresponding number $\delta > 0$ such that for all **x**

$$0 < |x - x_0| < \delta \;\Rightarrow\; f(x) < -B$$



P1:

Evaluate:

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 2x}$$

Solution:

$$\lim_{x \to 0} \frac{sin\ 4x}{sin\ 2x} = \lim_{x\to 0} \left[2.\frac{sin\ 4x}{4x} \cdot \frac{2x}{sin\ 2x}\right]$$

$$= 2 \,.\, \lim_{x\to 0}\left(\left[\frac{sin\ 4x}{4x}\right] \div \left[\frac{sin\ 2x}{2x}\right]\right)$$

Let $u = 4x$ and $v = 2x$. Now,$u \longrightarrow 0, v \longrightarrow 0$ as $x \longrightarrow 0$

$$= 2 \,.\, \lim_{u\to 0}\left[\frac{sin\ u}{u}\right] \div \lim_{v\to 0}\left[\frac{sin\ v}{v}\right]$$

$$= 2.1.1 = 2$$

P2:

$$\lim_{x\to 0}\frac{\sin x^2}{|x|}=$$

Solution:

$$\lim_{x\to 0}\frac{\sin x^2}{|x|}=$$

As $x \to 0, |x| \to 0^+ and\ |x|^2 \to 0^+$

$$\lim_{x\to 0}\frac{\sin x^2}{|x|}=\lim_{x\to 0}\frac{\sin|x|^2}{|x|}=\lim_{x\to 0}\frac{\sin|x|^2}{|x|^2}\times|x|$$

put $u = |x|^2$. Now, $u \longrightarrow 0^+$ as $x \longrightarrow 0$

$$=\left[\lim_{u\to 0^+}\frac{\sin u}{u}\right]\times\left[\lim_{x\to 0}|x|\right]=1\times 0=0\ \left(\text{since } \lim_{u\to 0}\frac{\sin u}{u}=1\ and\ \lim_{x\to 0}|x|=0\right.$$

P3:

$$\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2}{n^3} =$$

Solution:

$$\lim_{n \to \infty} \frac{1^2+2^2+3^2+4^2+\cdots+n^2}{n^3} \Rightarrow \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \Rightarrow \lim_{n \to \infty} \frac{\left(\frac{1}{n}+1\right)(2+\frac{1}{n})}{6}$$

$$\Rightarrow \lim_{n \to \infty} \frac{\left(\frac{1}{n}+1\right)(2+\frac{1}{n})}{6} \Rightarrow \lim_{n \to \infty} \frac{1 \times 2}{6} = \frac{1}{3}$$

P4:

$$\lim_{x \to 2^+} \frac{1}{x^2 - 4} =$$

Solution:

$$\lim_{x\to 2^+}\frac{1}{x^2-4}=$$

$$=\lim_{x\to 2^+}\frac{1}{(x-2)(x+2)}$$

$$=\lim_{x\to 2^+}\frac{1}{(x+2)}.\lim_{x\to 2^+}\frac{1}{(x-2)}$$

Let $x-2=u$ then $u\to 0^+ as\ x=2^+$

$$=\lim_{h\to 0}\frac{1}{(h+2+2)}.\lim_{h\to 0}\frac{1}{(h+2-2)}=\frac{1}{4}.\lim_{h\to 0}\frac{1}{h}=\infty$$

IP1:

$$\lim_{x \to 0} \frac{\tan x}{x}$$

Solution:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} . \lim_{x \to 0} \frac{1}{\cos x} = 1.1 = 1$$

IP2:

$$\lim_{x \to 0} \frac{\sin x^0}{x} =$$

Solution:

$$\lim_{x \to 0} \frac{\sin x^0}{x} =$$

We have to convert degrees into radians and

$$1^0 = \frac{\pi}{180} \text{ therefore, } x^0 = \frac{\pi x}{180}$$

Now, $\lim_{x \to 0} \frac{\sin x^0}{x} = \lim_{\left(\frac{\pi x}{180}\right) \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} \times \frac{\pi}{180} = 1 \times \frac{\pi}{180} = \frac{\pi}{180}$

IP3:

If $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$ with $a_n > 0, b_m > 0$ then show that $\lim_{x \to \infty} f(x) = \infty$ if $n > m$.

Solution:

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

$$\frac{x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)}{x^m\left(b_m + \frac{b_{m-1}}{x} + \cdots + \frac{m_1}{x^{m-1}} + \frac{b_0}{x^m}\right)} = x^{n-m}\frac{\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)}{\left(b_m + \frac{b_{m-1}}{x} + \cdots + \frac{m_1}{x^{m-1}} + \frac{b_0}{x^m}\right)}$$

As $x \to \infty$, all the quotients $\frac{a_{n-j}}{x^j}, \frac{b_{m-i}}{x^i}$ approach to zero

and $\lim_{x \to \infty} x^{n-m} = \infty$

$\therefore \lim_{x \to \infty} f(x) = \infty$

IP4:

$$\lim_{x\to 1^+}\frac{x}{x^2-1}=$$

Solution:

$$\lim_{x\to 1^+}\frac{x}{x^2-1}=$$

$$=\lim_{x\to 1^+}\frac{x}{(x-1)(x+1)}$$

$$=\lim_{x\to 1^+} x.\lim_{x\to 1^+}\frac{1}{(x+1)}.\lim_{x\to 1^+}\frac{1}{(x-1)}=\infty \qquad \left(\text{since } \lim_{x\to 1^+}\frac{1}{(x-1)}=\infty\right)$$

i) Find the limits of

a) $$\lim_{x \to 0} \frac{\sin x}{x\cos 2x}$$

b) $$\lim_{x \to a} \frac{\sin(x-a)}{(x^2-a^2)}$$

c) $$\lim_{x \to 0} \frac{\tan ax}{\sin bx}$$

d) $$\lim_{x \to 0} \frac{x+x\cos x}{\sin x.\cos x}$$

e) $$\lim_{x \to 1} \frac{\sin(x-1)}{(x^2-1)}$$

ii) Find the limits

a) $\lim_{x \to \infty} \frac{11x^3 - 3x^2 + 4}{13x^3 - 5x - 7}$

b) $\lim_{x \to \infty} \frac{3x^3 + 4x + 5}{2x^2 + 3x - 7}$

c) $\lim_{x \to -\infty} \frac{2x^2 - x + 3}{x^2 - 2x + 5}$

d) $\lim_{x \to -\infty} \frac{2x + 3}{\sqrt{x^2 - 1}}$

e) $\lim_{x \to \infty} \{\sqrt{x^2 + ax + b} - x\}$

f) $\lim_{x \to \infty} \frac{(3x - 1)(2x + 5)}{(x - 3)(3x - 7)}$

iii) Find the limits

a. $\lim_{x \to 0^+} \frac{1}{3x}$

b. $\lim_{x \to 2^-} \frac{3}{x-2}$

c. $\lim_{x \to -8^+} \frac{2x}{x+8}$

d. $\lim_{x \to 7} \frac{4}{(x-7)^2}$

e. $\lim_{x \to 0^+} \frac{2}{3x^{\frac{1}{3}}}$ $\quad \lim_{x \to 0^-} \frac{2}{3x^{\frac{1}{3}}}$

f. $\lim_{x \to 0} \frac{4}{x^{\frac{2}{5}}}$

g. $\lim_{x \to (\pi/2)^-} \tan x$

h. $\lim_{\theta \to 0^-} (1 + \csc \theta)$

6.1

Continuity at a Point

Learning objectives:

- To study the concept of continuity of a function at a point and to present continuity test
- To study the types of discontinuities through examples And
- To practice related problems

A continuous function is a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Several physical processes proceed continuously, and they are represented by functions of a real variable and have domains that are intervals or unions of separate intervals.

We study the continuity of a function at a point. There are three kinds of points to consider: ***interior points, left endpoint(s),*** and ***right endpoint(s)***.

Definition: continuity at a point

A function *f* is ***continuous at an interior point*** $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

Continuity at end points is defined by taking one-sided limits. A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

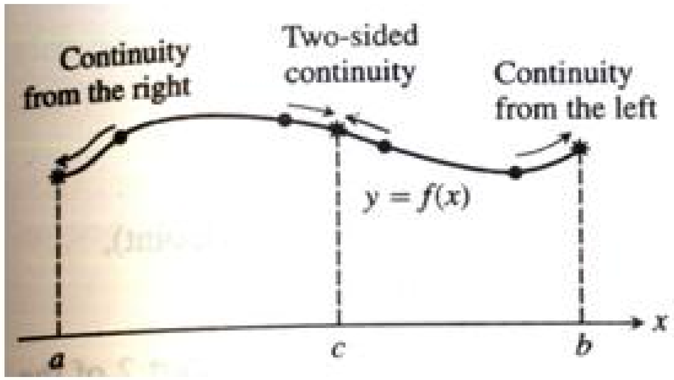
and **continuous at a right endpoint** $x = b$ of its domain if

$$\lim_{x \to b^-} f(x) = f(b)$$

In general, a function f is **right-continuous** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous** at c if $\lim_{x \to c^-} f(x) = f(c)$.

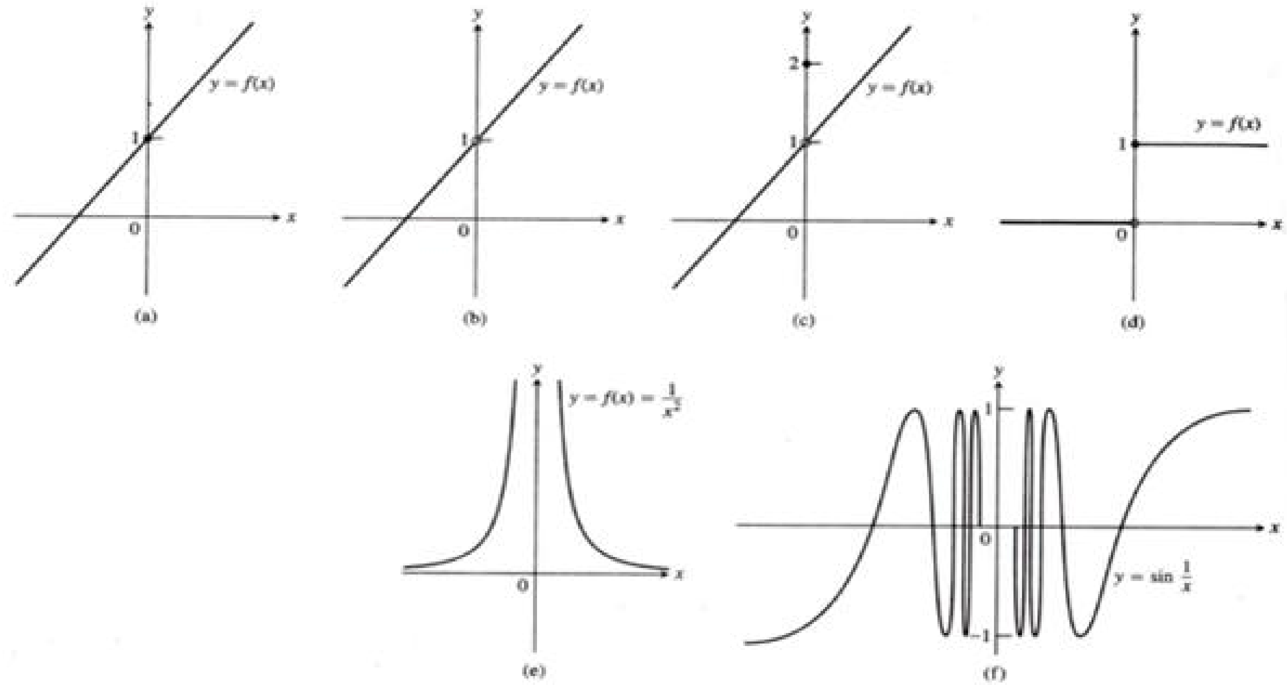
Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b.

A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c.



If a function f is not continuous at a point c, then we say that f is ***discontinuous*** at c and c is called a ***point of discontinuity*** of f.

Types of discontinuities:



The function in (a) is continuous at $x = 0$.

The function in (b) would be continuous if it had $f(0) = 1$.

The function in (c) would be continuous if $f(0)$ were 1 instead of 2.

The discontinuities in (b) and (c) are ***removable***. Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in parts (d) to (f) are of different nature: $\lim_{x\to 0} f(x)$ does not exist.

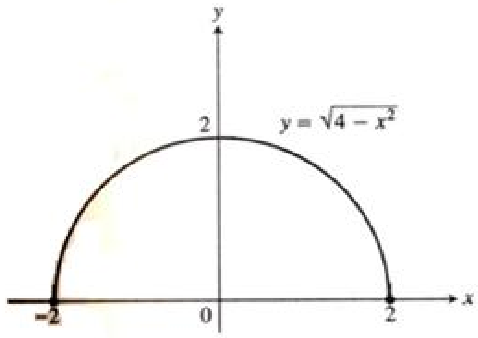
The step function in (d) has a **jump discontinuity**: the one-sided limits exist but have different values.

The function $f(x) = \frac{1}{x^2}$ in (e) has an **infinite discontinuity**.

These discontinuities are the ones most frequently encountered in applications. The function in (f) has an **oscillating discontinuity** at the origin because it oscillates too much to have limit as $x \to 0$.

Example*: A function continuous through out its domain.*

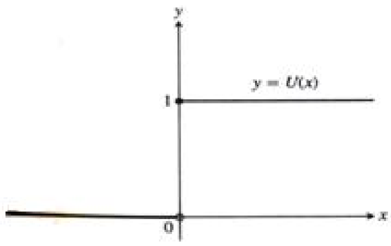
The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$.



This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.

Example*: The unit step function has jump discontinuity.*

The unit step function is graphed below.



It is right-continuous at $x = 0$, but is neither left-continuous there nor continuous at $x = 0$. It has a jump discontinuity at $x = 0$.

Continuity Test

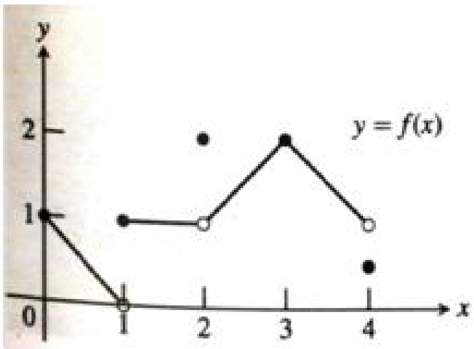
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

Example:

Consider the function $y = f(x)$, in the figure below, whose domain is the closed interval $[0, 4]$. Discuss the continuity of f at $x = 0, 1, 2, 3, 4$.



Solution:

f is continuous at $x = 0$ because $f(0)$ exists and $\lim_{x\to 0^+} f(x) = 1 = f(0)$.

f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist; f has different right- and left- hand limits at the interior point $x = 1$. However, f is right continuous at $x = 1$ because $f(1)$ exists, $\lim_{x\to 1^+} f(x) = 1$, and this equals the function value. Note that $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 0$. There fore $x = 1$ is a point of discontinuity and it is a jump discontinuity

f is discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) \neq f(2)$. Therefore $x = 2$ is a removable discontinuity, by setting $\lim_{x\to 2} f(x) = 1$.

f is continuous at $x = 3$ because $f(3)$ exists, $\lim_{x\to 3} f(x) = 2$, and this is equal to the function value.

f is discontinuous at the right endpoint $x = 4$ because $\lim_{x\to 4^-} f(x) \neq f(4)$.

P1:

Examine the continuity at $x = 0, 1, 2$ of the function f defined as

$$f(x) = \begin{cases} -x, & x \le 0 \\ 5x - 4, & 0 < x \le 1 \\ 4x^2 - 3x, & 1 < x < 2 \\ 3x + 4, & x \ge 2 \end{cases}$$

Solution:

The behavior of $f(x)$ at $x = 0$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5x - 4)$$
$$= \lim_{h\to 0}(5(0+h) - 4) = \lim_{h\to 0} 5h - 4 = -4$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x)$$
$$= \lim_{h\to 0} -(0-h) = \lim_{h\to 0} h = 0$$

$\Longrightarrow f(x)$ is discontinuous at $x = 0$ and $x = 0$ is a jump discontinuity of $f(x)$.

Continuity at $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x^2 - 3x)$$
$$= \lim_{h\to 0}(4(1+h)^2 - 3(1+h))$$
$$= \lim_{h\to 0}(4h^2 + 5h + 1) = 1$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x - 4)$$
$$= \lim_{h\to 0} 5(1-h) - 4 = 1$$

Further, $f(1) = 1$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\Longrightarrow f(x)$ is continuous at $x = 1$

Continuity at $x = 2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x + 4) = 10$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4x^2 - 3x) = 10$$

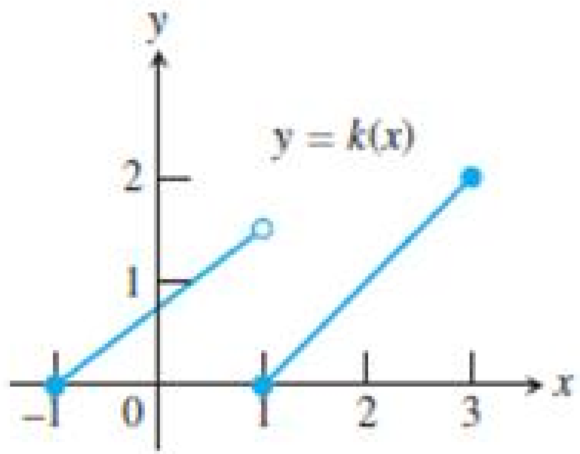
Further, $f(2) = 3(2) + 4 = 10$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$\Longrightarrow f(x)$ is continuous at $x = 2$

P2:

Discuss the continuity at $x = 1$ for the following:



Solution:

From the given graph $k(x)$ is defined at $x = 1$ and $k(1) = 0$

$$\lim_{x\to 1^-} k(x) = 1.5$$
$$\lim_{x\to 1^+} k(x) = 0$$
$$\lim_{x\to 1^-} k(x) \neq \lim_{x\to 1^+} k(x)$$

Thus, $k(x)$ is discontinuous at $x = 1$ and $x = 1$ is a jump discontinuity.

But $\lim_{x\to 1^+} k(x) = 0 = k(1)$. Therefore $k(x)$ is right continuous at $x = 1$.

P3:

Examine the continuity of

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$ at the origin.

Solution:

The given function is

$$f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 1, & x=0\end{cases}$$

Now, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{|x|}{x}$

$$=\lim_{h\to 0}\frac{|0+h|}{0+h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{|x|}{x}$$

$$=\lim_{h\to 0}\frac{|0-h|}{0-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

$\Longrightarrow f(x)$ is discontinuous at $x=0$ and $x=0$ is a jump discontinuity of $f(x)$.

P4:

If $f(x) = \frac{x^2-10x+25}{x^2-7x+10}$ for $x \neq 5$ and f is continuous at $x = 5$. Then find the value of $f(5)$.

Solution:

The Given function is

$$f(x) = \frac{x^2-10x+25}{x^2-7x+10} = \frac{(x-5)(x-5)}{(x-2)(x-5)} = \frac{(x-5)}{(x-2)}, x \neq 5$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{(x-5)}{(x-2)} = \lim_{h\to 0} \frac{(5+h-5)}{(5+h-2)} = 0$$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{(x-5)}{(x-2)} = \lim_{h\to 0} \frac{(5-h-5)}{(5-h-2)} = 0$$

$$\therefore \lim_{x\to 5} f(x) = 0$$

Given $f(x)$ is continuous at $x = 5$

$$\Rightarrow \lim_{x\to 5} f(x) = f(5) \Rightarrow 0 = f(5)$$

$$\therefore f(5) = 0$$

IP1:

The function f given by

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

Discuss the continuity at $x = 2$. Name the discontinuity if it is discontinuous at $x = 2$.

Solution:

The Given function is

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

Now, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2 - 8x^{-3})$

$= \lim_{h \to 0} (2 - 8(2+h)^{-3})$

$= \lim_{h \to 0} \left(2 - \frac{8}{8+12h+6h^2+h^3}\right) = 1$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{2}(x^2 - 4)$

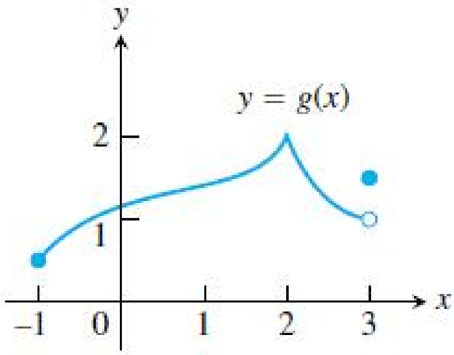
$= \lim_{h \to 0} \frac{1}{2}((2-h)^2 - 4)$

$= \lim_{h \to 0} \frac{1}{2}(-4h + h^2) = 0$

$\Longrightarrow f(x)$ is discontinuity at $x = 2$ and $x = 2$ is a jump discontinuity of $f(x)$.

IP2:

Discuss the continuity at the end points of the following:



Solution:

Step1:

Notice that $g(-1)$ exists, and $g(-1) = 0.5$ and

$\lim_{x\to -1^+} g(x) = 0.5$.

Now $\lim_{x\to -1^+} g(x) = g(-1)$. Since $x = -1$ is the left end point, $g(x)$ is continuous at $x = -1$.

Step2:

Now, $\lim_{x\to 3^-} g(x) = 1$ and $g(3) = 1.5$

$\therefore \lim_{x\to 3^-} g(x) \neq g(3)$

Therefore $g(x)$ is discontinuous at $x = 3$

(since $x = 3$ is left end point)

Note:

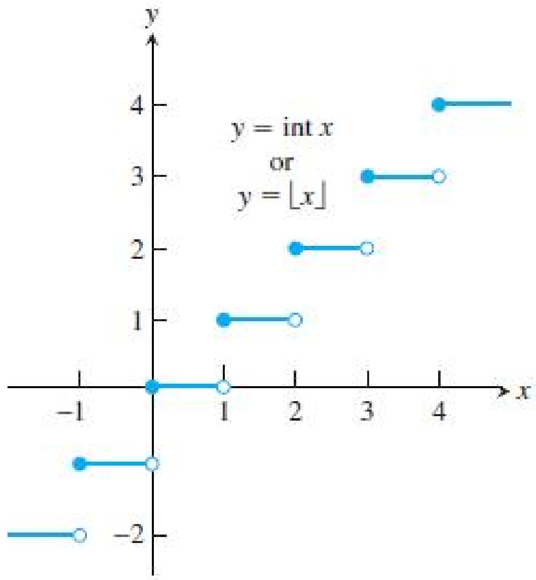
$x = 3$ is a removable discontinuity by redefining $g(3)$ as $\lim_{x\to 3^-} g(x)$ $i.e.,$ 1.

IP3:

Discuss the continuity of $f(x) = \lfloor x \rfloor$

Solution:

The given function is $f(x) = \lfloor x \rfloor$. $y = \lfloor x \rfloor$ is graphed below



$f(x)$ is discontinuous at every integer because the limit does not exists at an integer n

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1$$

And

$$\lim_{x \to n^+} \lfloor x \rfloor = n$$

So the left hand limit and right hand limit are not equal as $x \to n$.

Since $\lfloor n \rfloor = n$, the greatest integer function is right continuous at every integer n.

Therefore, the greatest integer function is continuous at every real number other than the integers.

IP4:

If $\boldsymbol{f(x) = \frac{x^2-9}{x^2-2x-3}}$ for $\boldsymbol{x \neq 3}$ and $\boldsymbol{f(x)}$ is continuous at $\boldsymbol{x = 3}$.

Then find the value of $\boldsymbol{f(3)}$.

Solution:

Step1:

The given function is

$$f(x) = \frac{x^2-9}{x^2-2x-3} = \frac{(x+3)(x-3)}{(x+1)(x-3)} = \frac{(x+3)}{(x+1)}, x \neq 3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{(x+3)}{(x+1)} = \lim_{h\to 0} \frac{(3+h+3)}{(3+h+1)} = \frac{3}{2}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(x+3)}{(x+1)} = \lim_{h\to 0} \frac{(3-h+3)}{(3-h+1)} = \frac{3}{2}$$

$$\therefore \lim_{x\to 3} f(x) = \frac{3}{2}$$

Given $f(x)$ is continuous at $x = 3$

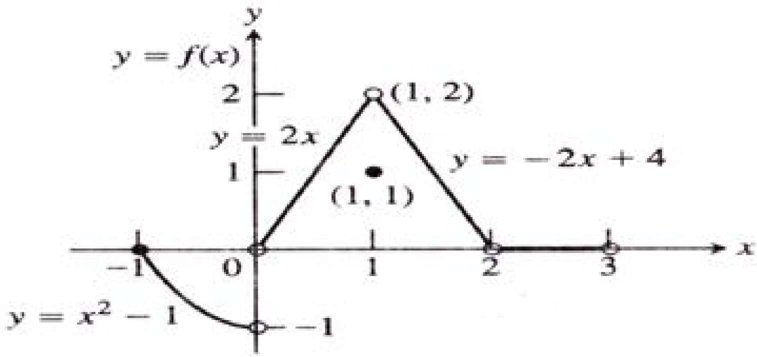
$$\Longrightarrow \lim_{x\to 3} f(x) = f(3) \Longrightarrow \frac{3}{2} = f(3)$$

$$\therefore f(3) = \frac{3}{2}$$

1. The function

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0<x<1 \\ 1 & x=1 \\ -2x+4 & 1<x<2 \\ 0 & 2<x<3 \end{cases}$$

is graphed below.



a. Does $f(-1)$ exist?

b. Does $\lim_{x\to 1^+} f(x)$ exist?

c. Does $\lim_{x\to 1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

e. Is f defined at $x = 2$?

f. Is f continuous at $x = 2$?

2. Is function f defined by $f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous at $x = 0$.

3. Check the continuity of f given by

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 0 \\ x - 5 & \text{if } 0 \leq x \leq 1 \\ 4x^2 - 9 & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$$ at the points 0,1 and 2.

4. If f is function defined by $f(x) = \begin{cases} \dfrac{x-1}{\sqrt{x}-1} & \text{if } x > 1 \\ 5-3x & \text{if } -2 \le x \le 1 \\ \dfrac{6}{x-10} & \text{if } x < -2 \end{cases}$ then

discuss the continuity of f.

5. If $f(x) = \frac{x^2 - 1}{x - 1}$. Discuss the continuity at $x = 1$.

6. At what points are the functions continuous?

a. $y = \frac{1}{x-2} - 3x$

b. $y = \frac{x+1}{x^2 - 4x + 3}$

c. $y = |x-1| + \sin x$

d. $y = \frac{\sin x}{x}$

e. $y = \csc 2x$

f. $y = \frac{x \tan x}{x^2 + 1}$

g. $y = \sqrt{2x+3}$

h. $y = (2x-1)^{1/3}$

6.2

Rules of Continuity

Learning objectives:

- To state the properties of continuous functions.
- To study the continuity of polynomials, rational functions, absolute value function and trigonometric functions.
- To define the continuous extension of a function to a point.

  And
- To practice related problems.

Algebraic combinations of continuous functions are continuous wherever they are defined

Theorem: Continuity of Algebraic Combinations

If functions *f* and *g* are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf , where k is any number
4. $\frac{f}{g}$, provided $g(c) \neq 0$
5. $\left(f(x)\right)^{\frac{m}{n}}$, m and n are integers, $n \neq 0$.

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

Theorem: Continuity of Polynomials and Rational Functions

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

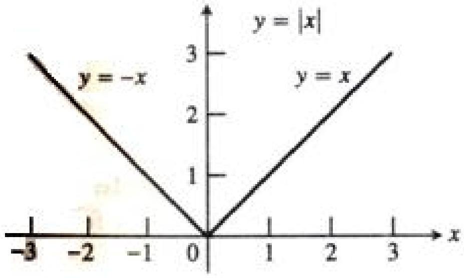
Example: The functions $f(x) = x^4 + 20$ and $g(x) = 5x(x-2)$ are continuous at every value of x. The function

$$r(x) = \frac{x^4+20}{5x(x-2)}$$

is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0.

Example: Continuity of $f(x) = |x|$

The function $f(x) = |x|$ is continuous at every value of x.



If $x > 0$, we have $f(x) = x$ is a polynomial.
If $x < 0$, we have $f(x) = -x$ is another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$.

Example:

We will later show that the functions $\sin x$ and $\cos x$ are continuous at every value of x. It then follows that the quotients

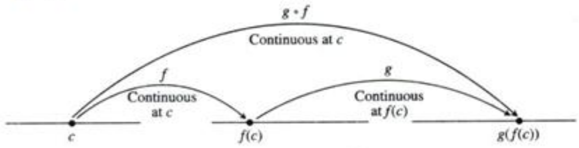
$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined.

Continuity of Composites:

Theorem:

If f is continuous at c, and g is continuous at $f(c)$, then the composite gof is continuous at c.



The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied.

Example: The following functions are continuous everywhere on their respective domains.

a) $y = \sqrt{x}$

b) $y = \sqrt{x^2 - 2x - 5}$

c) $y = \frac{x\cos\left(x^{\frac{2}{3}}\right)}{1+x^4}$

d) $y = \left|\frac{x-2}{x^2-2}\right|$

Continuous Extension to a Point

If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } \ x = c \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension** of f to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.

Example:

Show that $f(x) = \frac{x^2+x-6}{x^2-4}$ has a continuous extension to $x = 2$, and find that extension.

Solution:

Notice that $f(2)$ is not defined. If $x \neq 2$, we have

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$. It is continuous at $x = 2$, having the value of $5/4$. Thus F is the continuous extension of f to $x = 2$. The graphs of f and F are shown below.

P1:

If the functions f and g are continuous at $x = c$ then $f \pm g$ are continuous at $x = c$.

Proof:

Given f and g are continuous functions at $x = c$

$i.e.$, $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

Now, $\lim_{x\to c}(f+g)(x) = \lim_{x\to c}\big(f(x)+g(x)\big)$

$= \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(sum rule for limits)

$= f(c) + g(c)$

(since f, g are continuous at c)

$= (f+g)(c)$

Therefore, $f + g$ is continuous at $x = c$.

Similarly $f - g$ is continuous at $x = c$.

Hence the result

P2:

Every rational function is continuous wherever it is defined.

Proof:

Let the rational function be $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Let c be any real number such that $q(c) \neq 0$.

Then,

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} \quad \text{(Quotient rule for limits)}$$

$$= \frac{p(c)}{q(c)}$$

(since polynomials are continuous everywhere)

Therefore, $\frac{p(x)}{q(x)}$ is continuous at $x = c$. Thus every rational function is continuous wherever it is defined.

Hence the result

P3:

Prove that the function $f(x) = \left|1 - x + |x|\right|$ is continuous everywhere.

Solution:

The given function is $f(x)=\left|1-x+|x|\right|$

Define g by $g(x)=1-x+|x|$ and h by $h(x)=|x|$, for all real x.

Now ,

$$\begin{aligned}(hog)(x) &= h\big(g(x)\big) \\ &= h(1-x+|x|) \\ &= \left|1-x+|x|\right| \\ &= f(x)\end{aligned}$$

Since $|x|$ is a continuous function for all x, $h(x)$ is continuous everywhere.

Since $g(x)$ is a sum of a polynomial function and the modulus function, $g(x)$ is continuous everywhere.

Since $f(x)$ is a composite of two everywhere continuous functions $g(x)$ and $h(x)$; $f(x)$ is continuous everywhere.

P4:

Show that $h(t) = \frac{t^2+3t-10}{t-2}$ has a continuous extension to $t = 2$, and find that extension.

Solution:

The given function is $h(t) = \frac{t^2+3t-10}{t-2}$. Notice that $h(t)$ is not defined at $t = 2$. If $t \neq 2$, we have

$$h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t + 5$$

Let $H(t) = t + 5$ and $H(t) = h(t)$ at $t \neq 2$.

Since $H(t)$ is a polynomial; $H(t)$ is continuous at $t = 2$ and has the value 7 at $t = 2$.

Thus, $H(t)$ is the continuous extension of $h(t)$ to $t = 2$ and $\lim_{t\to 2} h(t) = 7$

IP1:

Every polynomial is continuous at every point c of the real line.

Proof:

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_i \epsilon R$, $i = 0,1,2, \ldots, n$, $a_n \neq 0$ be any polynomial. Let c be any real number. We know that $\lim_{x\to c} x = c$ and $\lim_{x\to c} k = k$.

Therefore,

$$\lim_{x\to c} p(x) = \lim_{x\to c}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$$

$$= \lim_{x\to c} a_0 + a_1 \lim_{x\to c} x + a_2 (\lim_{x\to c} x)^2 + \cdots + a_n (\lim_{x\to c} x)^n$$

(By the properties of limits)

$$= a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n$$

$$= p(c)$$

Thus $p(x)$ is continuous at every $c \in R$.

Hence the result

IP2:

Prove that $\sin x$, $\cos x$ are continuous everywhere.

Proof:

We first note that

$$\lim_{x\to 0} \sin x = 0 \text{ and } \lim_{x\to 0} \cos x = 1$$

Let $f(x) = \sin x$. Let c be any real number. Then,

$$\begin{aligned}
\lim_{x\to c^+} f(x) &= \lim_{h\to 0} f(c+h) \\
&= \lim_{h\to 0} \sin(c+h) \\
&= \lim_{h\to 0} \sin c \cos h + \lim_{h\to 0} \cos c \sin h \\
&= \sin c \lim_{h\to 0} \cos h + \cos c \lim_{h\to 0} \sin h \\
&= \sin c
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to c^-} f(x) &= \lim_{h\to 0} f(c-h) \\
&= \lim_{h\to 0} \sin(c-h) \\
&= \lim_{h\to 0} \sin c \cos h - \lim_{h\to 0} \cos c \sin h \\
&= \sin c \lim_{h\to 0} \cos h - \cos c \lim_{h\to 0} \sin h \\
&= \sin c
\end{aligned}$$

$\Longrightarrow \lim_{x\to c^+} \sin x = \lim_{x\to c^-} \sin x = \sin c$

$\Longrightarrow \lim_{x\to c} \sin x = \sin c$. Therefore, $\sin x$ is continuous at c.

Thus, $\sin x$ is continuous everywhere (since c is any real number).

Similarly $\cos x$ is continuous everywhere.

IP3:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Solution:

Step1:

The given function is $f(x) = |\cos x|$

Step2:

Define g by $g(x) = \cos x$ and h by $h(x) = |x|$

$$\begin{aligned}(hog)(x) &= h(g(x)) \\ &= h(\cos x) \\ &= |\cos x| \\ &= f(x)\end{aligned}$$

Step3:

Notice that h and g are continuous everywhere. Since f is the composite of two functions g and h; f is continuous everywhere.

IP4:

Show that $g(x) = \frac{x^2-16}{x^2-3x-4}$ has a continuous extension to $x = 4$. Find that extension.

Solution:

Step1:

The given function is $g(x) = \frac{x^2-16}{x^2-3x-4}$

Step2:

At $x = 4$, $g(x)$ is not defined. If $x \neq 4$ we have

$$g(x) = \frac{x^2-16}{x^2-3x-4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1}$$

Let, $G(x) = \frac{x+4}{x+1}$ and $G(x) = g(x)$ at $x \neq 4$

Since $G(x)$ is a rational function with denominator not vanishing at $x = 4$; $G(x)$ is continuous at $x = 4$ and has the value $\frac{8}{5}$ at $x = 4$.

Step3:

Thus, G is the continuous extension of g to $x = 4$ and

$\lim_{x\to4} \frac{x^2-16}{x^2-3x-4} = \lim_{x\to4} g(x) = \frac{8}{5}$

1. Find the limits. Are the functions continuous at the point being approached?

a. $\lim_{x\to\pi} \sin(x - \sin x)$

b. $\lim_{y\to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

c. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$

d. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$

e. $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{\frac{1}{3}}\right)\right)$

f. $\lim_{x\to\frac{\pi}{6}} \sqrt{\cos^2 x + 5\sqrt{3}\tan x}$

2. Find the continuous extension of

a. $g(x) = \frac{x^2-9}{x-3}$ to $x = 3$

b. $f(s) = \frac{s^3-1}{s^2-1}$ to $s = 1$

c. $g(x) = \frac{x^2-16}{x^2-3x-4}$ to $x = 4$

d. $h(t) = \frac{(t^2+3t-10)}{t-2}$ to $t = 2$

e. $f(x) = \frac{x^2-9}{x^2-x-12}$ to $x = -3$

3. For what value of a is $f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$

continuous at every x.

4. Given that the function f defined by

$$f(x) = \begin{cases} 2x - 1 & if\ x > 2 \\ k & if\ x = 2 \\ x^2 - 1 & if\ x < 2 \end{cases}$$ is continuous at every x. Then find the value of k.

5. For what values of b is $g(x) = \begin{cases} x & if\ x < -2 \\ bx^2 & if\ x \geq 2 \end{cases}$

continuous at every x.

6. Prove that the function $f(x) = \sqrt{1 + \sqrt{2x + 1}}$ is continuous at $x = 2$.

7. Prove that the function $g(x) = \cos(3t + 4)$ is continuous at every real number.

6.3
Continuity on Intervals

Learning objectives:

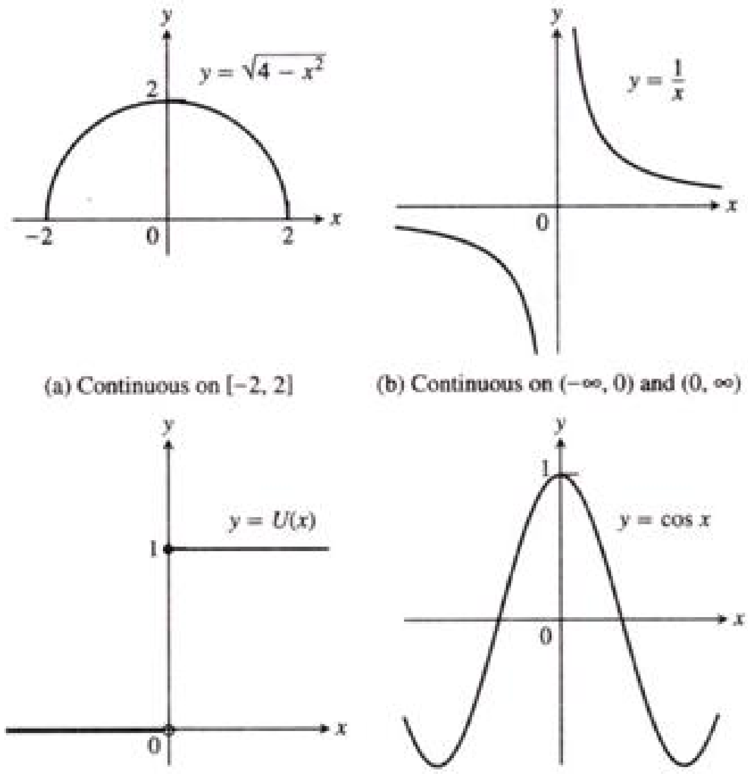
- To define continuity of a function on its domain.
- To study intermediate value theorem and its application to assert the existence of a zero of a function.
  And
- To practice the related problems.

A function is called **continuous** if it is continuous every where in its domain.
A function that is not continuous throughout its entire domain may be continuous when restricted to particular intervals within the domain.

A function *f* is said to be ***continuous on an interval*** I in its domain if $\lim_{x\to c} f(x) = f(c)$ at every interior point c and if the appropriate one-sided limits equal the function values at the endpoints.
A function continuous on an interval I is automatically continuous on any interval contained in I.

Polynomials are continuous on every interval, and rational functions are continuous on every interval on which they are defined.

Example: Functions continuous on intervals



(a) Continuous on [−2, 2] (b) Continuous on (−∞, 0) and (0, ∞)

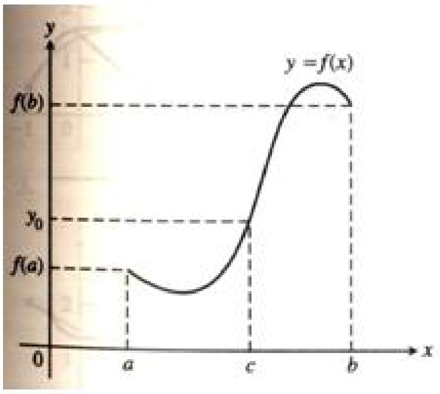
(c) Continuous on (−∞, 0) and [0, ∞) (d) Continuous on (−∞, ∞)

Functions that are continuous on intervals have properties that make them particularly useful in applications. One of these is the intermediate value property.

A function is said to have the ***intermediate value property*** if whenever it takes on two values, it also takes on all the values in between.

Theorem: The Intermediate Value Theorem

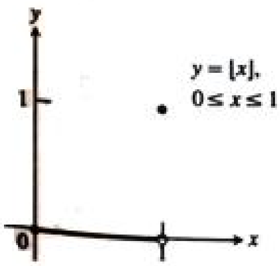
Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.



The function f, being continuous on $[a, b]$, takes on every value between $f(a)$and $f(b)$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system.

The continuity of $f$on I is essential to the theorem. If $f$is discontinuous even at one point of f, the theorem does not apply. For example, it will not apply for the function graphed below.



The function $f(x) = [x], 0 \leq x \leq 1$, does not take on any value between $f(0) = 0$ and $f(1) = 1$.

The above theorem is the reason for the graph of a function continuous on an interval I cannot have any breaks. It will be connected, a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like the graph of the greatest integer function $[x]$ or separate branches like the graph of $\frac{1}{x}$.

We call a solution of the equation $f(x) = 0$ a **root** or **zero** of the function f. The Intermediate Value Theorem tells the following:

If f is continuous, then any interval on which f changes sign must contain a zero of the function.

Example: Is any real number exactly 1 less than its cube?

Solution: Any such number must satisfy the equation

$$x = x^3 - 1$$

$$i.e., \ x^3 - x - 1 = 0$$

Hence we are looking for zeros of $f(x) = x^3 - x - 1$. By trial, we find that $f(1) = -1$ and $f(2) = 5$. Then, by the Intermediate Value Theorem, there is at least one number in [1,2] where *f* is zero. The answer to the question is then "yes".

P1:

$$\text{If } f(x) = \begin{cases} -2\sin x, & \text{for } -\pi \le x \le \frac{-\pi}{2} \\ a\sin x + b, & \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$$

is continuous on $[-\pi, \pi]$, then find the values of a and b.

Solution:

Since f is continuous on $[-\pi, \pi]$, it is continuous at $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$

Now, f is continuous at $x = \frac{-\pi}{2}$

$\Rightarrow \lim_{x \to (-\frac{\pi}{2})^-} f(x) = \lim_{x \to (-\frac{\pi}{2})^+} f(x) = f(-\frac{\pi}{2})$

$\Rightarrow \lim_{x \to -\frac{\pi}{2}} (-2 \sin x) = \lim_{x \to -\frac{\pi}{2}} (a \sin x + b)$

$\Rightarrow -2 \sin(-\frac{\pi}{2}) = a \sin(-\frac{\pi}{2}) + b$

$\Rightarrow 2 = -a + b = 2$

$\Rightarrow -a + b = 2$ ....(1)

Again f is continuous at $x = \frac{\pi}{2}$

$\Rightarrow \lim_{x \to (\frac{\pi}{2})^-} f(x) = \lim_{x \to (\frac{\pi}{2})^+} f(x) = f(\frac{\pi}{2})$

$\Rightarrow \lim_{x \to \frac{\pi}{2}} (a \sin x + b) = \lim_{x \to \frac{\pi}{2}} \cos x = \cos\frac{\pi}{2}$

$\Rightarrow a \sin\frac{\pi}{2} + b = \cos\frac{\pi}{2}$

$\Rightarrow a + b = 0$ ....(2)

Solving equations (1) and (2), we get $a = -1, b = 1$

P2:

Examine the continuity in the interval $(-\infty, \infty)$ of the function defined as follows

$$f(x) = \begin{cases} 2 & , \quad if \ x \in (-\infty, 0) \\ 1 + \cos x & , \quad if \ x \in \left[0, \frac{\pi}{2}\right) \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & , \quad if \ x \in \left[\frac{\pi}{2}, \infty\right) \end{cases}$$

Solution:

When $x \in (-\infty, 0)$, we have $f(x) = 2$. It is a constant function and so is continuous on $(-\infty, 0)$.

When $x \in \left[0, \frac{\pi}{2}\right)$, we have $f(x) = 1 + \cos x$. Clearly, it is continuous on its domain.

When $x \in \left[\frac{\pi}{2}, \infty\right)$, we have $(x) = 2 + \left(x - \frac{\pi}{2}\right)^2$. Since it is a polynomial, it is continuous on $\left(\frac{\pi}{2}, \infty\right)$.

Therefore, the function f is continuous everywhere except possibly at $= 0, \frac{\pi}{2}$.

Firstly we consider $x = 0$

Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1 + \cos x) = \lim_{h\to 0}(1 + \cos(0 + h))$$
$$= \lim_{h\to 0}(1 + \cos h) = 1 + 1 = 2$$
$$(\because \lim_{h\to 0} \cos h = 1)$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 = 2$$

Further, $f(0) = 1 + \cos 0 = 1 + 1 = 2$

$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

Thus, f is continuous at $x = 0$.

Now, we consider $x = \frac{\pi}{2}$, we have

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^+}\left(2 + \left(x - \frac{\pi}{2}\right)^2\right)$$
$$= \lim_{h\to 0}\left(2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2\right)$$
$$= \lim_{h\to 0}(2 + h^2) = 2$$
$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}(1 + \cos x)$$
$$= \lim_{h\to 0}\left(1 + \cos\left(\frac{\pi}{2} - h\right)\right)$$
$$= \lim_{h\to 0}(1 + \sin h)$$
$$= 1 + 0 = 1 \qquad (\because \lim_{h\to 0} \sin h = 0)$$
$$\therefore \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) \neq \lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x)$$

Thus, f is discontinuous at $x = \frac{\pi}{2}$.

Therefore, f is continuous on $\left(-\infty, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \infty\right)$

P3:

Show that the function $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Solution:

Let $f(x) = x^3 - 15x + 1$. Clearly $f(x)$ is continuous on $[-4,4]$.

Now, $f(-4) = (-4)^3 - 15(-4) + 1 = -3 < 0$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15 > 0$$

$$f(1) = (1)^3 - 15(1) + 1 = -13 < 0$$

$$f(4) = (4)^3 - 15(4) + 1 = 5 > 0$$

By the intermediate value theorem, $f(x) = 0$ for some x in each intervals $(-4,-1)$, $(-1,1)$ and $(1,4)$.

$i.e.,\ \ x^3 - 15x + 1 = 0$ has three solutions in $[-4,4]$

P4:

The function f is defined by $f(x) = 2x^3 - 5x^2 - 10x + 5$.

Prove that $f(x)$ has at least one zero in $[0, 1]$.

Solution:

The Given function is $f(x) = 2x^3 - 5x^2 - 10x + 5$.

Since all polynomial functions are continuous everywhere;

$f(x)$ is continuous every where.

By trial we see

$$f(0) = 2(0)^3 - 5(0)^2 - 10(0) + 5 = 5 > 0$$
$$f(1) = 2(1)^3 - 5(1)^2 - 10(1) + 5 = -8 < 0$$

By the intermediate value theorem there exists a number c in $(0,1)$ such that $f(c) = 0$. Thus, the function $f(x)$ has at least one zero in $[0,1]$.

IP1:

If $f(x) = \begin{cases} x^2 + ax + b & \text{for } 0 \le x < 2 \\ 3x + 2 & \text{for } 2 \le x \le 4 \\ 2ax + 5b & \text{for } 4 < x < 8 \end{cases}$

is continuous on $[0, 8]$, then find the values of a and b.

Solution:

Step1:

Since f is continuous on $[0,8]$, it is continuous at $x = 2$ and $x = 4$

Step2:

Now, for $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(x^2 + ax + b) = \lim_{x\to 2^+}(3x + 2) = 3(2) + 2$$

$$\Rightarrow 4 + 2a + b = 8$$

$$\Rightarrow 2a + b = 4 \ldots\ldots\ldots\ldots\ldots(1)$$

Step3:

Now for $x = 4$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$

$$\lim_{x\to 4^-}(3x + 2) = \lim_{x\to 4^+}(2ax + 5b) = 3(4) + 2$$

$$\Rightarrow 8a + 5b = 14 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Step4:

Solving (1) and (2), we get $a = -1$, $b = 6$.

IP2:

Show that the function

$$f(x) = \begin{cases} x^2, & if\ 0 \le x < 1 \\ 1, & if\ 1 \le x < \sqrt{2} \\ \frac{2}{x^2}, & if\ \sqrt{2} \le x < \infty \end{cases}$$ is continuous on $[0, \infty)$.

Solution:

Step1:

When $0 \le x < 1$, we have $f(x) = x^2$. Since it is a polynomial, it is continuous on $[0,1)$.

When $1 \le x < \sqrt{2}$, we have $f(x) = 1$. Since it is a constant function, it is continuous on $(1, \sqrt{2})$.

When $\sqrt{2} \le x < \infty$, we have $f(x) = \frac{2}{x^2}$. Since it is a rational function and $x^2 \ne 0$ in the defined domain, it is continuous on $(\sqrt{2}, \infty)$.

We have to prove $f(x)$ is continuous on $[0, \infty)$ it is enough to prove f is continuous at $x = 1$ and $x = \sqrt{2}$.

Step2:

At $x = 1$,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2$$
$$= \lim_{h\to 0}(1+h)^2 = 1$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} 1 = 1$$

Further, $f(1) = 1$

$$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

Thus, f is continuous at $x = 1$.

Step3:

At $x = \sqrt{2}$,

$$\lim_{x\to(\sqrt{2})^+} f(x) = \lim_{x\to(\sqrt{2})^+} \left(\frac{2}{x^2}\right)$$
$$= \lim_{h\to 0} \frac{2}{(\sqrt{2}+h)^2} = 1$$
$$\lim_{x\to(\sqrt{2})^-} f(x) = \lim_{x\to(\sqrt{2})^-}(1) = 1$$

Further, $f(\sqrt{2}) = 1$

$$\therefore \lim_{x\to(\sqrt{2})^-} f(x) = \lim_{x\to(\sqrt{2})^-} f(x) = f(\sqrt{2})$$

Thus, f is continuous at $x = \sqrt{2}$.

Step4:

Therefore, f is continuous on $[0, \infty)$

IP3:

Show that the function $\boldsymbol{F(x) = (x-a)^2(x-b)^2 + x}$ takes on the value $\frac{(a+b)}{2}$ for some value $\boldsymbol{x}$.

Solution:

Step1:

The given function is: $F(x) = (x-a)^2(x-b)^2 + x$.

Without loss of generality, assume that $a < b$.

Then, $F(a) = a$ and $F(b) = b$

Step2:

By the intermediate value theorem we have: If f continuous on $[a, b]$, then it takes every value between $f(a)$ and $f(b)$.

Since $a < \frac{a+b}{2} < b$, there is a number c between a and b such that $F(c) = \frac{a+b}{2}$.

IP4:

Prove that the function $f(x) = x^3 + x^2 - 4$ has at least one zero in the interval $(1, 2)$.

Solution:

Step1:

The given function is: $f(x) = x^3 + x^2 - 4$.

Since all polynomial functions are continuous everywhere, $f(x)$ is continuous everywhere.

Step2:

Now,

$$f(1) = 1^3 + 1^2 - 4 = -2 < 0$$
$$f(2) = 2^3 + 2^2 - 4 = 8 > 0$$

Step3:

Therefore, by intermediate value theorem there exists a number c in $(1,2)$ such that $f(c) = 0$.Thus, the function $f(x)$ has at least one zero in $(1,2)$.

1. If the function f, defined by

$$f(x)=\begin{cases} kx+1 & if -\infty \le x \le 1 \\ x^2-1 & if \quad 1 \le x \le \infty \end{cases}$$ is a continuous function

on $[-1,2]$. Then find the value of k.

2. Find the values of a and b so that the function $f(x)$ defined

by $$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , if\ 0 \leq x < \frac{\pi}{4} \\ 2x\cos x + b & , if\ \frac{\pi}{2} \leq x < \frac{\pi}{2} \\ a\cos 2x - b\sin x & , if\ \frac{\pi}{2} \leq x < \pi \end{cases}.$$

3. Discuss the continuity of $f(x) = \sqrt{16 - x^2}$.

4. Discuss the continuity of the function $f(x) = \sqrt{\frac{x-2}{5-x}}$ in the interval $(2,5)$.

5. Find the interval in which the function $f(x) = \sqrt{x} + \sqrt{2x - 1}$ is continuous.

6. Explain why the equation $x - \cos x = 0$ has at least one solution.

7. Show that $P(x) = 2x^3 - 5x^2 - 10x + 5$ has a zero some where between -1 and 2.

8. Show that $x^3 - x + 1 = 0$ has a zero in the interval $[-2,0]$

6.4

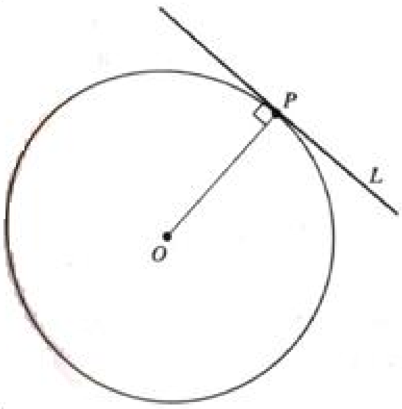
Tangent Lines

Learning objectives:

- To define the tangent to a curve at a point on the curve and to find it.
  And
- To practice the related problems.

Tangent to a Curve

From the geometry, we know the tangents to circles. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P. Such a line just *touches* the circle.

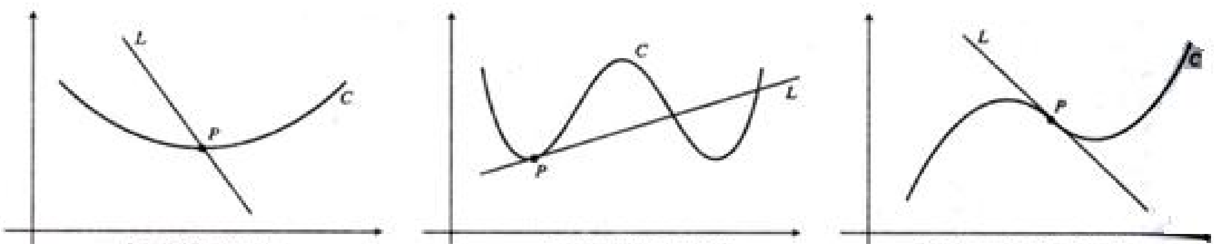


The following statements are valid.

1. L passes through P and is perpendicular to the line from P to the center of C.

1. L passes through P and is perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.

These statements may not apply consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency.

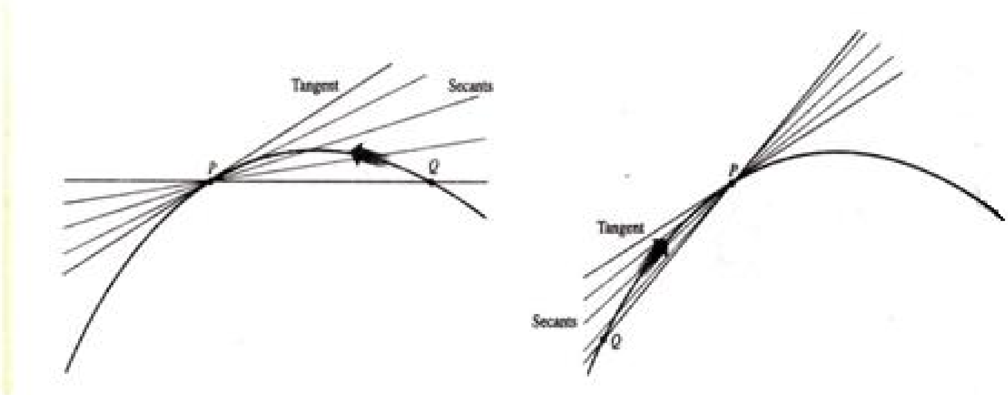


L meets *C* only at *P* but is not tangent to *C*.

L is tangent to *C* at *P* but meets *C* at several points.

L is tangent to *C* at *P* but lies on two sides of *C*, crossing *C* at *P*.

To define tangency for general curves, we take into account the behavior of the secants through P and nearby points Q (on C) as Q moves toward P along the curve.



The procedure is as follows:

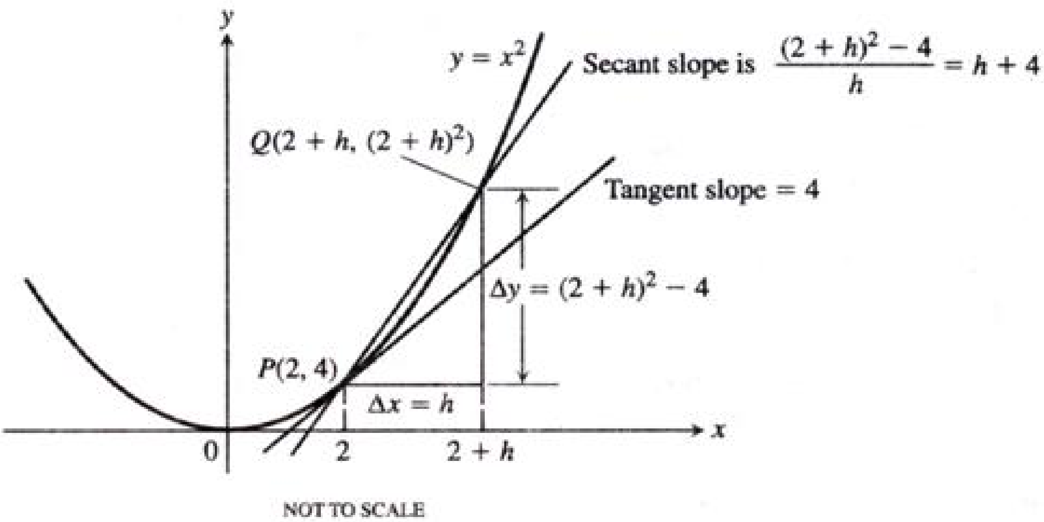
1. We calculate the slope of the secant PQ
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, we take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

Example: Find the slope of the parabola $y = x^2$ at the point P(2,4). Write an equation for the tangent to the parabola at this point.

Solution: Consider the secant line through P(2,4) and Q$(2+h, (2+h)^2)$ nearby.

Secant slope $= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h + 4$

If $h > 0$, Q lies above and to the right of P, as in the figure below.



As Q approaches P along the curve, *h* approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} h + 4 = 4$$

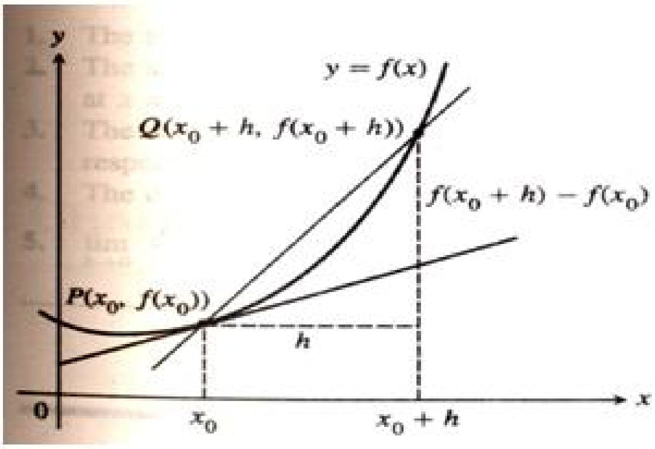
We take 4 to be the parabola's slope at P. The tangent to the parabola at P is the line through P with slope 4. The equation of the tangent to the parabola at P is,

$$y = 4 + 4(x-2) \qquad \text{Point-slope equation}$$

$$\Rightarrow y = 4x - 4$$

Finding a Tangent to the Graph of a function

We use the same procedure to find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$. We calculate the slope of the secant through P and a point Q$(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$.



The tangent slope is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

Definitions

The **slope of the curve** $y = f(x)$ at the point P$(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example

a) Find the slope of the curve $y = \frac{1}{x}$ $at\ x = a \neq 0$

b) Where does the slope equal $-1/4$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) We have $f(x) = \frac{1}{x}$. The slope at $\left(a, \frac{1}{a}\right)$ is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-h}{ha(a+h)}$$

$$= \frac{-1}{a^2}$$

b) Given the slope is $-1/4$. Therefore,

$$\frac{-1}{a^2} = \frac{-1}{4} \quad \Rightarrow \quad a^2 = 4 \quad \Rightarrow a = 2, -2$$

The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$.

Rates of Change

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}$$

is called the **difference quotient of *f* at x_0**. If the difference quotient has a limit as *h* approaches zero, that limit is called the **derivative of *f* at x_0**.

If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, the derivative gives the function's rate of change with respect to x at the point $x = x_0$.

Example

A rock falls from the top of a $50\ m$ cliff. Physical experiments show that a solid object dropped from the rest to fall freely near the surface of the earth will be $y = 5t^2\ m$ during the first t sec.

What is the rock's speed at $t = 1$ sec?

Solution:

$$f(t) = 5t^2$$

The rock's speed at the instant $t = 1$ sec is

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{5(1+h)^2-5(1)^2}{h}$$

$$= \lim_{h\to 0}\frac{5(h^2+2h)}{h} = \lim_{h\to 0} 5(h+2)$$

$$= 10\ m/sec$$

We note that the following statements refer to the same thing:

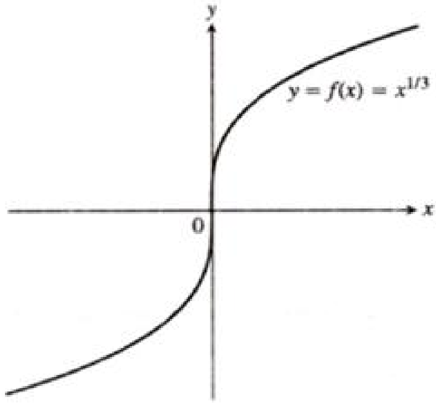
1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

Vertical tangents:

We say that the curve $y = f(x)$ has a ***vertical tangent*** at the point $x = x_0$ if

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \infty \text{ or } -\infty$$

1) Consider the function $y = f(x) = x^{\frac{1}{3}}$. Its graph is shown below.

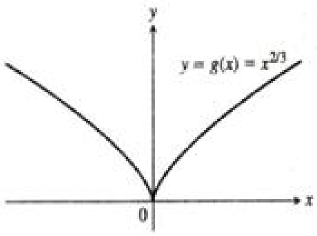


At $x = 0$:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(h)^{\frac{1}{3}}-0}{h} = \lim_{h\to 0} \frac{1}{(h)^{\frac{2}{3}}} = \infty$$

So, there is a vertical tangent at $x = 0$.

2) Consider the function $y = g(x) = x^{\frac{2}{3}}$. Its graph is shown below.



At $x = 0$:

$$\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{(h)^{\frac{2}{3}}-0}{h} = \lim_{h\to 0} \frac{1}{(h)^{\frac{1}{3}}}$$

The limit does not exist, because the limit is ∞ from the right and $-\infty$ from the left.

So, there is no vertical tangent at $x = 0$.

P1:

Find the slope of the curve $y = 1 - x^2$ at $x = 2$.

Solution:

The given curve is: $y = 1 - x^2$. At $x = 2$ we have $y = 1 - 2^2 = -3$. Then the point $P(2, -3)$ is on the given curve.

We know that the slope of the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The slope of the curve $y = 1 - x^2$ at the point $P(2, -3)$ is

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(1 - (2+h)^2) - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{-h(4+h)}{h}$$

$$= \lim_{h \to 0} -(4 + h) = -4$$

Slope of the given curve at $x = 2$ is -4.

P2:

Find the equation for the tangent line to the curve $y = (x - 1)^2 + 1$ at the point $P(1, 1)$.

Solution:

The given curve is: $y = (x-1)^2 + 1$. The point $P(1,1)$ is on the curve.

We know that the slope of the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the tangent to the curve $y = (x-1)^2 + 1$ at the point $P(1,1)$ is

$$m = \lim_{h\to 0} \frac{[(1+h-1)^2+1] - [(1-1)^2+1]}{h}$$

$$= \lim_{h\to 0} \frac{h^2+1-1}{h} = \lim_{h\to 0} h = 0$$

Equation of the tangent line to the given curve at $P(1,1)$ having slope $m = 0$ is

$$(y-1) = 0(x-1) \implies y - 1 = 0$$

P3:

Find the equation of the straight line having slope $\frac{1}{4}$ that is tangent to the curve $y = \sqrt{x}$.

Solution:

Given that the slope of the tangent to the curve $y = \sqrt{x}$ is $m = \frac{1}{4}$.

We know that slope of the tangent to the curve $y = f(x)$ is at $(x, \sqrt{x})$

$$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \Rightarrow \frac{1}{4} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$

Now, $x = 4 \Rightarrow y = \sqrt{4} = 2$ and the equation of the tangent line at the point $(4,2)$ and having slope $\frac{1}{4}$ is

$$y - 2 = \frac{1}{4}(x - 4) \Rightarrow x - 4y + 4 = 0$$

P4:

What is the rate of change of the volume of a ball$(V = \frac{4}{3}\pi r^3)$ with respect to the radius when radius is $r = 2$.

Solution:

Volume of the ball $V = f(r) = \frac{4}{3}\pi r^3$

We know that the rate of change of $f(x)$ with respect to x at $x = x_0$ is

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The rate of change of $f(r)$ with respect to the radius r at $r = 2$ is

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(2+h)^3 - \frac{4}{3}\pi(2)^3}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(8+h^3+6h^2+12h-8)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(h^3+6h^2+12h)}{h}$$

$$= \lim_{h \to 0} \frac{4}{3}\pi(h^2 + 6h + 12)$$

$$= \frac{4}{3}\pi(12) = 16\pi$$

The rate of change of the volume of a ball with respect to the radius when radius is $r = 2$ is 16π

IP1:

Find the slope of curve $\boldsymbol{y = \frac{x-1}{x+1}}$ at $\boldsymbol{x = 0}$.

Solution:

Step1:

The given curve is: $y = \frac{x-1}{x+1}$. At $x = 0$ we have, $y = \frac{0-1}{0+1} = -1$.

Now, the point $P(0, -1)$ is on the given curve.

Step2:

We know that the slope of the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the curve $y = \frac{x-1}{x+1}$ at the point $P(0, -1)$ is

$$m = \lim_{h \to 0} \frac{\left(\frac{h-1}{h+1}\right) - 1}{h}$$

$$= \lim_{h \to 0} \frac{(h-1)-(h+1)}{h(h+1)}$$

$$= \lim_{h \to 0} \frac{2h}{h(h+1)} = \lim_{h \to 0} \frac{2}{h+1} = 2$$

Step3:

Slope of the curve given curve at $x = 0$ is 2.

IP2:

Find the equation for the tangent line to the curve $y = \frac{1}{x^2}$ at the point $P(-1, 1)$.

Solution:

Step1:

The given curve is: $y = \frac{1}{x^2}$. The point $P(-1,1)$ is on the curve.

Step2:

We know that the slope of the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the tangent to the curve $y = \frac{1}{x^2}$ at the point $P(-1,1)$ is

$$m = \lim_{h\to 0} \frac{\frac{1}{(-1+h)^2} - \frac{1}{(-1)^2}}{h}$$

$$= \lim_{h\to 0} \frac{-(-2h+h^2)}{h(-1+h)^2}$$

$$= \lim_{h\to 0} \frac{2-h}{(h-1)^2} = 2$$

Step3:

Equation of the tangent line to the given curve at $P(-1,1)$ with slope $m = 2$ is

$$y + 1 = 2(x - 1) \Rightarrow 2x - y + 3 = 0$$

IP3:

Find the equation of the straight line having slope 2 that is tangent to the curve $y = x^2 - 2x + 3$.

Solution:

Step1:

Given that the slope of the tangent to the curve $y = x^2 - 2x + 3$ is $m = 2$

Step2:

We know that slope of the tangent to the curve $y = f(x)$ at $(x, x^2 - 2x + 3)$ is

$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \quad 2 = \lim_{h\to 0} \frac{[(x+h)^2-2(x+h)+3]-[x^2-2x+3]}{h}$$

$$\Rightarrow \quad 2 = \lim_{h\to 0} \frac{[x^2+2xh+h^2-2x-2h+3]-[x^2-2x+3]}{h}$$

$$\Rightarrow \quad 2 = \lim_{h\to 0} \frac{2xh-2h+h^2}{h}$$

$$\Rightarrow \quad 2 = \lim_{h\to 0} 2x + h - 2 \Rightarrow \quad 2 = 2x + 2$$

$$\Rightarrow \quad x = 0$$

Now, $x = 0 \ \Rightarrow \ y = 0^2 - 2(0) + 3 = 3$

Step3:

The equation of the straight line at the point $(0,3)$ and having slope $m = 2$ is

$$y - 3 = 2(x - 0) \ \Rightarrow \ 2x - y + 3 = 0$$

IP4:

Verify whether the curve $y = x^{\frac{2}{3}} - (x-1)^{\frac{1}{3}}$ has a vertical tangent at $x = 1$.

Solution:

The given curve is: $y = x^{\frac{2}{3}} - (x-1)^{\frac{1}{3}}$.

We know that the curve $y = f(x)$ has a vertical tangent at $x = x_0$ if

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \infty \text{ or } -\infty$$

Vertical tangent at $x = 1$:

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^{\frac{2}{3}}-(1+h-1)^{\frac{1}{3}}-1}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^{\frac{2}{3}}-1}{h} - \lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h}$$

Now, $\lim_{h\to 0}\frac{(1+h)^{\frac{2}{3}}-1}{h} = \lim_{h\to 0}\left[\frac{(1+h)^{\frac{2}{3}}-1}{h} \times \frac{\left((1+h)^{\frac{4}{3}}+(1+h)^{\frac{2}{3}}+1\right)}{\left((1+h)^{\frac{2}{3}}+(1+h)^{\frac{2}{3}}+1\right)}\right]$

$$= \lim_{h\to 0}\left[\frac{(1+h)^2-1}{h\left((1+h)^{\frac{2}{3}}+(1+h)^{\frac{2}{3}}+1\right)}\right]$$

$$= \lim_{h\to 0}\left[\frac{h(h+2)}{h\left((1+h)^{\frac{2}{3}}+(1+h)^{\frac{2}{3}}+1\right)}\right] = \frac{2}{3}$$

$$\therefore \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \frac{2}{3} - \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}} = \infty \quad \left(\because \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}} = \infty\right)$$

Therefore, the given curve has a vertical tangent at $x = 1$.

1. Find the slope of the curve at the point indicated.

a. $y = 5x^2$, $x = -1$

b. $y = \frac{1}{x-1}$, $x = 3$

c. $y = 3x^4 - 4x$, $x = 4$

d. $y = x^3 - x + 1$, $x = 2$

e. $y = x^3 - 3x + 2$, $x = 3$

2. Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

a. $f(x) = x^2 + 1, \quad (2,5)$

b. $g(x) = \frac{x}{x-2}, \quad (3,3)$

c. $h(t) = t^3, \quad (2,8)$

d. $f(x) = \sqrt{x}, \quad (4,2)$

e. $f(x) = x - 2x^2, \quad (1,-1)$

f. $g(x) = \frac{8}{x^2}, \quad (2,2)$

g. $h(t) = t^3 + 3t, \quad (1,4)$

h. $f(x) = \sqrt{x+1}, \quad (8,3)$

3. Find an equation for the tangent to the curve at the given point.

a. $y = 4 - x^2$ $(-1,3)$

b. $y = 2\sqrt{x}$ $(1,2)$

c. $y = x^3$ $(-2,-8)$

d. $y = \frac{1}{x^3}$ $\left(-2,-\frac{1}{8}\right)$

4. At what points do the graph of the function has horizontal tangents?

a. $f(x) = x^2 + 4x - 1$

b. $f(x) = x^3 - 3x$

5.

a. Does the graph of

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

b. Does the graph of

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

has a vertical tangent at the point $(0,1)$? Give reasons for your answer.

c. Does the graph of

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

has a vertical tangent at the origin? Give reasons for your answer.

6. Verify whether the curve has a vertical tangent at the point indicated.

a. $y = x^{\frac{2}{5}}, \qquad x = 0$

b. $y = x^{\frac{1}{5}}, \qquad x = 0$

c. $y = 4x^{\frac{2}{5}} - 2x, \qquad x = 0$

d. $y = x^{\frac{1}{3}} + (x - 1)^{\frac{1}{3}}, \quad x = 1$

7. Find equations of all lines having slope −1 that are tangent to the curve $y = 1/(x-1)$.

8. An object is dropped from the top of a 100-m-high tower. Its height aboveground after t seconds is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

9. At t sec after liftoff, the height of a rocket is $3t^2$ft. how fast the rocket climbing 10sec after liftoff?

10. What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to its radius when the radius is $r = 3$?